APPLICATIONS OF CONFORMAL FIELD THEORY AND STRING THEORY IN STATISTICAL SYSTEMS

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Abstract

This thesis investigates different aspects of conformal field theory and string theory and their applications in statistical properties of systems. First, we study the free fermions in planar Ising model and its scaling limit at criticality. On the one hand, we examine the relation between the transfer matrix formalism and discrete holomorphicity. We show that the fermion operators of the Ising model satisfy a complexification of the defining relations of s-holomorphicity, a strong notion of discrete holomorphicity, and examples of fermion correlation functions are shown to reproduce s-holomorphic parafermionic observables. On the other hand, we study the relation between fermionic conformal field theory and Schramm Loewner evolution by focusing on the interfaces and fermionic correlation functions of the Ising model. We demonstrate an explicit, rigorous realization of the CFT/SLE correspondence in the case of Ising model. Second, we develop a statistical framework for bosonic string theory in order to study transport properties of black holes in the context of membrane paradigm. We find that the shear viscosity of a highly excited bosonic string is equal to that of black hole horizon up to a numerical factor.

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In this thesis

This thesis contains an introductory part and the following articles:

[i] C. Hongler, K. Kytölä, A. Zahabi, Discrete holomorphicity and Ising model operator formalism (2012), arXiv: 1211.7299 [math-ph]

[ii] A. Zahabi, Vertex operator algebra, conformal field theory and stochastic Loewner evolution in Ising model (2013)

[iii] Y. Sasai, A. Zahabi, Shear viscosity of a highly excited string and black hole membrane paradigm, Phys. Rev. D 83,026002 (2011) arXiv: 1010.5380 [hep-th]

Introduction

Contents

1 OVERVIEW

The conformal field theory is one of the most successful theories describing facets of nature from low energies to high energies. It provides illuminating explanations and various applications in wide spectrum of systems from statistical lattice models to string theory. The powerful techniques of CFT enable us to predict and calculate definite properties and exact results of different systems. In fact, CFTs are solvable toy models of more complicated interacting quantum field theories and recently it has been conjectured that some specific exactly solvable two-dimensional CFTs describe higher dimensional interacting quantum field theories. For a short and general review on different aspects of CFT consult [FRS10].

However, the rigorous derivations and proofs of CFT results are not available completely but in the case of $2d$ CFTs, they have been gradually obtained by the new mathematical techniques from different branches of mathematics such as analysis, combinatoric and probability. One of the aims of this thesis is to shed light on these new relations and clarify them through explicit concrete examples. We will employ the $2d$ CFT in two different directions, namely two applications of fermionic and bosonic conformal field theories: i) free fermion of Ising model and its relation to discrete holomorphicity and Schramm Loewner evolution and ii) transport properties of string theory and black holes in membrane paradigm. The basic underlying theory of both topics is conformal field theory; i) fermionic conformal field theory in continuum limit of the critical Ising model and ii) bosonic conformal field theory for bosonic strings. Let us discuss two topics separately.

I) Ising model is one of the simplest models of statistical mechanics which has very interesting critical behavior and rich mathematical structure. This model of ferromagnetism was introduced in the 1920's and solved in the one-dimensional case by Ising. In 1940's, the thermodynamic limit of the model was shown to exhibit a phase transition between ferromagnetic and paramagnetic phases when the space dimension is two or above [McWu73]. The two-dimensional Ising model, on which this thesis also focuses, has particularly remarkable mathematical properties such as exact solvability: in the thermodynamic limit, the free energy per unit volume can be calculated explicitly as a function of the parameters of the model.

The exact solvability of the Ising model discovered by Onsager in 1944 is based on infinite dimensional (conformal) symmetry inherent in the two-dimensions. This is the first example of infinite-dimensional algebra which has been used in physics. After Onsager discovery, several other methods of exact solutions were found. Kaufman in 1949, [Kau49], used the free fermions for computation of free energy. Other methods include the combinatorial ones used by Kac, Ward, Hurst and Green and star-triangle equations and functional equations used by Baxter [Ba08]. In addition, in the fermionic formalism explicit calculations of free energy and some important correlation functions can be performed. Such solutions of the Ising model were mostly based on the transfer matrix formalism.

At the critical point of the phase transition, conformal invariance emerges in the scaling limit of the model. This allows to compute some correlation functions by only symmetry considerations. The conformal invariance was observed and studied extensively in the physics literature of conformal field theories since 1980's, specially after seminal papers by Belavin, Polyakov and Zamolodchikov [BPZ84a] and [BPZ84b]. However, conformal invariance in conformal field theory approach is not clearly stated in mathematical sense and the procedure to take the scaling limit is non-rigorous. In recent years remarkable progress has been made in rigorous understanding of the conformal invariance properties.

The recent advances in rigorous understanding of conformal invariance started with making clear probabilistic formulations of the conformal invariance property in simply connected sub-domains of the plane. Most notably, these formulations lead to the introduction of the random fractal curves known as Schramm-Loewner evolutions. A successful strategy, put forward especially by Smirnov, was to show first the conformal invariance of the scaling limit of a single observable or a correlation function, and then use this knowledge to obtain conformal invariance of the scaling limit of random curves appearing in the model. The proof that an observable of the lattice model converges to a conformally invariant scaling limit required a suitable and strong enough notion of complex analysis on the lattice. For the Ising model, the correct notion of discrete holomorphicity is now termed s-holomorphicity, [Smi06], [Smi10b].

In another direction, the relation between SLE curves and observables on the one hand and conformal field theory on the other hand has been grown, during last years. Bauer and Bernard, [BaBe06], have made a neat observation that the differential equations of conformal field theory and stochastic differential equations of SLE are related. Moreover, they constructed an operator formalism approach to study SLE. These observation and construction led to a correspondence between CFT and SLE. In addition, many realizations of CFT/SLE correspondence such as relations between SLE and Gaussian free field, [KaMa11], Coulomb gas formalism, [Gr06], etc. have been investigated.

II) Two-dimensional bosonic CFT or theory of free scalar fields is one of the most studied theories in physics and it is known as Gaussian free fields in mathematics [KaMa11]. This theory has been applied extensively from statistical field theory to high energy physics. A field theory living on the $2d$ world-sheet of the bosonic strings describing the symmetries and dynamics of open and closed strings is an example of two-dimensional bosonic CFT. This model of CFT is employed successfully in different areas of particle physics and cosmology. As one of these examples, we present the long-standing unsolved problems of theoretical physics namely, the statistical and thermodynamical properties of black holes, which are tackled by string theory.

The source of these puzzles returns to an observation in early 1970; the laws of black hole dynamics resemble the laws of thermodynamics. For example, area law of the black hole which is that the area of the horizon can not decrease during the black hole evolution, is similar to the entropy law of the thermodynamics. These similarities and relations among the others led physicists to associate the temperature and entropy to black holes, known as Hawking temperature and Bekenstein-Hawking entropy. Moreover, there is another subtle puzzle called information paradox. This paradox is inherent in the fact that the black hole radiates as black-body with Hawking temperature. This radiation contains no information and therefore, the information in the structure of the in-falling matter is destroyed during the evaporation of the black hole, [To97].

String theory as a theory with rich physical and mathematical structure can provide microscopic suggestions and descriptions to solve these puzzles, [Se95] and [CaMa96]. In fact, the correspondence between string theory and black holes has been studied extensively during the last twenty five years and some aspects of this correspondence such as Bekenstein-Hawking entropy formula for black holes have been obtained by string theory techniques [StVa96]. In order to study the transition and matching between black hole and strings some formalism has been proposed such as string/black hole correspondence by Horowitz and Polchinski, [HoPo97] and [HoPo98]. Basically, when the curvature of space-time becomes of the order of string scale, the entropies of a highly excited string and black hole match.

Another unsolved puzzle of the black hole physics which is one of the subjects of this thesis is the membrane paradigm. In membrane paradigm, we think of the extended horizon of a black hole as a fictitious fluid with hydrodynamical, electromechanical and thermal properties. As we explained, the string theory at some regimes describes the black hole physics and its paradoxes. Our proposal in this thesis extends this correspondence and tries to explain the black hole membrane paradigm via string theory.

In the following, we will discuss two parts of the thesis, separately.

PART I:

2 TWO-DIMENSIONAL CONFORMAL FIELD THEORY AND STOCHASTIC LOEWNER EVOLUTION

In this chapter we review basic backgrounds and standard results of conformal field theory (CFT) in physics language, [DMS96], [BlPl09]. This review will be useful for proper understanding of two applications in this part and also the next part. This chapter includes the most fundamentally important and necessary aspects of conformal field theory in the Euclidean plane $\mathbb{R}^2$ or equivalently, the complex plane $\mathbb{C}$. First, we briefly summarize the definitions of conformal group and algebra. Second, the representation of conformal group, namely the conformal fields and their correlation functions and differential equations are reviewed. Then we continue with two basic examples of CFTs, the bosonic and fermionic free fields. In the last part of this chapter, we will briefly review the standard constructions and results in stochastic Loewner evolution (SLE). Moreover, we will describe some standard aspects of the relation between CFT and SLE in details.

GLOBAL CFT IN D-DIMENSIONS

From a general point of view, quantum field theories are constructed based on symmetry groups. In fact, they are invariant under Poincar$\acute{e}$ transformations or their Euclidean version. Furthermore, one can ask also for more general symmetry groups and the quantum field theories as the representations of those. A possible general extension of the Poincare group is the conformal group which is a group of transformations that preserve angles but not necessarily lengths. The CFTs are theories that behave nicely under conformal transformations, specially they are scale invariant theories. In other words, the conformal transformations in D-dimensional space-time are special type of coordinate transformations that preserve the metric $g_{\mu\nu}(x)$ up to a scale change,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x), \tag{1}$$

where $\mu, \nu = 0, 1, ..., D-1$. Notice that the $\Omega(x) = 1$ case presents the Poincar$\acute{e}$ group as a subgroup of conformal group.

As we mentioned, conformal transformations are extension of Poincar$\acute{e}$ transformations. In addition to Poincar$\acute{e}$ transformations, there are dilation or scale transformations and special conformal transformations which is the conjugation of the translation by an inversion. Therefore, we have a collection of transformations; i) translations and rotations which preserve angles as well as lengths and ii) scale transformations and special conformal transformations which preserve the angles only. These transformations can be written in the following form:

$$\begin{aligned}
x^\mu &\to x^\mu + a^\mu, \ \textit{translations}, \\
x^\mu &\to \omega^\mu_\nu x^\nu, \ \textit{rotations}, \\
x^\mu &\to \lambda x^\mu, \ \textit{dilations}, \\
x^\mu &\to \frac{x^\mu + a^\mu x^2}{1 + 2a\cdot x + a^2 x^2}, \ \textit{special conformal transformations},
\end{aligned} \tag{2}$$

where a^μ is an arbitrary constant vector, ω^μ_ν is an anti-symmetric rotation matrix, λ is a scalar and the dot product represents contraction with metric, $x^2 = x \cdot x = g_{\mu\nu} x^\mu x^\nu$. Respectively, the generators of these transformations are $P_\mu, J_{\mu\nu}, N$ and K_μ, given by

$$P_\mu = -\mathrm{i}\partial_\mu, \quad J_{\mu\nu} = \mathrm{i}(x_\mu \partial_\nu - x_\nu \partial_\mu), \quad N = -\mathrm{i}x\cdot\partial, \quad K_\mu = -\mathrm{i}[x^2\partial_\mu - 2x_\mu(x\cdot\partial)]. \tag{3}$$

In $D-$dimensional manifold, the number of these generators is $D + \frac{1}{2}D(D-1) + 1 + D = \frac{1}{2}(D+2)(D+1)$ which is the same as number of generators of the group of rotations in $D+2$ dimensions. The conformal transformations form a group which is called conformal group.

Two-dimensional CFT

The aim of this section is to introduce the powerful method of conformal field theory in two-dimensions. Conformal symmetries in two-dimensions are of great importance because of their infinite dimensional symmetry algebra. Thus, the $2d$ CFTs are exactly solvable by symmetry consideration. However, the applications of the two-dimensional conformal field theory are so vast, from statistical physics to string theory. Therefore, introducing the general framework of the $2d$ CFT is useful and beneficial in the sense that it can be applied to many different systems with slightly different formulations and interpretations.

Having said this brief motivation, we will start by physical definitions of conformal field theory in a descriptive way and then we will focus on the theoretical basis and properties of the $2d$ CFT and its bosonic and fermionic realizations. Although the conformal transformations can be considered in any dimensions but we will consider a special case of $D = 2$. In the two-dimensional case, it is easier to work with complex coordinates $z, \bar{z}$.

2.1 Conformal group and algebra

As we mentioned, the *conformal group* consists of globally invertible conformal transformations, but in fact, local conformal transformations in two-dimensions are analytic coordinate transformations by locally invertible holomorphic and anti-holomorphic functions f and $\bar{f}$,

$$z \to f(z), \quad \bar{z} \to \bar{f}(\bar{z}). \tag{4}$$

The infinitesimal conformal transformations are given by the infinitesimal forms of the functions, $f(z) = z + \epsilon(z)$, $\bar{f}(\bar{z}) = \bar{z} + \bar{\epsilon}(\bar{z})$, in which $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ can be expanded by the Laurent expansions,

$$\epsilon(z) = -\sum_n \epsilon_n z^{n+1}, \qquad \bar{\epsilon}(\bar{z}) = -\sum_n \bar{\epsilon}_n \bar{z}^{n+1}. \tag{5}$$

The generators of these infinitesimal conformal transformations are $l_n = -z^{n+1}\frac{\partial}{\partial z}$ and $\bar{l}_n = -\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}$. In fact, they are generators of a Lie algebra which is called *conformal or Witt algebra*. The Witt algebra is a complex Lie algebra of meromorphic vector fields on a circle. The vector fields are expanded by the generators $l_n, \bar{l}_n$ and the generators satisfy the Witt algebra commutators

$$[l_n, l_m] = (n-m)l_{n+m}, \qquad [\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m}, \qquad [l_n, \bar{l}_m] = 0. \tag{6}$$

Notice that the number of generators l_n is infinite and therefore, the algebra of infinitesimal two-dimensional Euclidean conformal transformations is infinite dimensional.

The conformal algebra (6) is not well-defined even on the *Riemann sphere* $S^2 = \mathbb{C} \cup \{\infty\}$, because a vector field $v(z) = -\sum_n a_n l_n$, has singularities at $z \to 0$ and $z \to \infty$, unless $a_n = 0$ for $n < -1$ and $n > 1$. Therefore, globally defined and invertible transformations are those which are generated by $l_{0,\pm 1}$ and $\bar{l}_{0,\pm 1}$. The generators $l_{-1}, \bar{l}_{-1}$ generate translations, $\mathrm{i}(l_0 - \bar{l}_0)$ generates rotations, $(l_0 + \bar{l}_0)$ generates dilatations and $l_1, \bar{l}_1$ generate special conformal transformations.

In the finite form, these transformations are of the following form,

$$z \to \frac{az+b}{cz+d}, \tag{7}$$

where $a, b, c, d \in \mathbb{C}$, $ad - bc = 1$. These transformations form a group which is called *Möbius group* and it is isomorphic to restricted conformal group, $SL(2,\mathbb{C})/\mathbb{Z}_2 \cong SO(3,1)$. The quotient by $\mathbb{Z}_2$ reflects the invariance of the transformations under replacement of the parameters a, b, c, d by their negatives.

The Witt algebra can be interpreted as a symmetry algebra of classical conformal theory. However, the appropriate symmetry algebra of quantum conformal theory is the central extension of the Witt algebra and it is called Virasoro algebra $\mathfrak{Vir}_c$. The generators of the Virasoro algebra satisfy

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{C}{12}(n^3 - n)\delta_{n+m,0}, \qquad [L_n, C] = 0, \tag{8}$$

where C is the central charge c times the unit operator. The central charge refers to the quantum breaking of the classical conformal symmetry or in other word, *conformal anomaly*.

2.2 Conformal fields and their properties

In a naive sense, conformal field theories are representations of conformal algebra and its corresponding conformal transformations. In a conformally invariant field theory, the *primary* field operators $\phi_i(z_i, \bar{z}_i)$ with real-valued conformal weights $(h_i, \bar{h}_i)$ are defined by following transformation rule

$$\phi_i(z_i, \bar{z}_i) = (\frac{\partial f}{\partial z_i})^{h_i}(\frac{\partial \bar{f}}{\partial \bar{z}_i})^{\bar{h}_i}\phi_i(f(z_i), \bar{f}(\bar{z}_i)), \tag{9}$$

for all conformal transformations $f, \bar{f}$ as defined in previous section. The mode expansion of a holomorphic primary field $\phi_i(z_i)$ on the plane can be obtained by Fourier expansion of the field on a cylinder and then transform it to the plane by using the above relation,

$$\phi_i(z_i) = \sum_{n\in\mathbb{Z}} \phi_{i_n} z_i^{-n-h_i}. \tag{10}$$

If the conformal transformation is of the special type of restricted conformal group, $f \in SL(2, \mathbb{C})/\mathbb{Z}_2$, then the fields, defined by eq. (9), are called *quasi-primary* fields. The field contents of the conformal field theory consist of primary and the fields which are called *secondary* or *descendant*. In a naive sense, the secondary fields are obtained by taking the derivatives and products of primary fields. We will see that the descendant fields are obtained from the primary fields in a rather complicated way. We separate the primary and quasi-primary fields from rest of the fields because they have rather simple transformation rules.

An important example of the quasi-primary fields is the *stress-tensor* field. In general, stress tensor is defined by variation of the action with respect to the metric. In terms of the partition function it can be written as response of the system to a local change in the geometry, $T^{\mu\nu}(x) \propto (\delta \ln Z)/(\delta g_{\mu\nu(x)})$. Invariance of the theory under translation, rotation and scale transformations imply that stress tensor is conserved, $\partial_\mu T^{\mu\nu}(x) = 0$ and it is traceless, $T^\mu_\mu = 0$. These equations in two-dimensional Euclidean space translate to $T_{z\bar{z}} = T_{\bar{z}z} = 0$ and $\partial_{\bar{z}} T_{zz} = \partial_z T_{\bar{z}\bar{z}} = 0$ and therefore, stress tensor has two non-vanishing components, a *chiral* and an *anti-chiral* fields,

$$T_{zz}(z, \bar{z}) = T(z), \qquad T_{\bar{z}\bar{z}}(z, \bar{z}) = \bar{T}(\bar{z}). \tag{11}$$

It can be shown that the conformal transformation of the stress tensor is

$$T(z) = T(f(z))f'(z)^2 + \frac{c}{12}S(f(z), z), \tag{12}$$

where $f'(z)$ is derivative of f with respect to z and $S(f(z), z) = (\frac{f'''(z)}{f'(z)}) - \frac{3}{2}(\frac{f''(z)}{f'(z)})^2$ is called *Schwarzian derivative*. The relation between generators of infinitesimal conformal transformations L_n and $T(z)$ is defined by

$$L_n = \oint \frac{dz}{2\pi \mathrm{i}} z^{n+1} T(z). \tag{13}$$

Consequently, the Virasoro algebra generators appear as the modes in the formal power expansion of holomorphic stress tensor $T(z)$,

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \tag{14}$$

We have similar relations for $\bar{L}_n$ and $\bar{T}$. Having defined primary and quasi-primary operator fields, we need to define products of operator fields.

Operator product expansion

The essential idea of operator product expansion is to replace the product of two local operator fields at different points in space-time z and w, with a series of other local operator fields at either one of the initial points z or w, times c-number coefficient functions which depend on $(z-w)$. More precisely, the product of the field operators $\phi_i(z,\bar{z})$ and $\phi_j(w,\bar{w})$ which has singularities in the limits $z \to w$ and $\bar{z} \to \bar{w}$ satisfies the algebra called operator product expansion (OPE),

$$\phi_i(z,\bar{z}).\phi_j(w,\bar{w}) = \sum_k c_{ijk}(z-w)^{-h_i-h_j+h_k}(\bar{z}-\bar{w})^{-\bar{h}_i-\bar{h}_j+\bar{h}_k}\phi_k(z,\bar{z}) + ..., \tag{15}$$

where the sum is over complete set of operators ϕ_k indexed by k, c_{ijk} is called structure constant and ... are non-singular terms in the product, the terms that are not divergent in the limit $z \to w$ and $\bar{z} \to \bar{w}$. For simplicity, let us consider holomorphic fields and some important OPEs of them such as OPE of the stress tensor $T(z)$ and a primary field $\phi_i(z_i)$, which produces a set of secondary fields as follow

$$T(z)\phi_i(z_i) = \frac{h_i}{(z-z_i)^2}\phi_i(z_i) + \frac{1}{(z-z_i)}\partial_{z_i}\phi_i(z_i) + \phi_i^{(-2)}(z_i) + (z-z_i)\phi_i^{(-3)}(z_i) + ..., \tag{16}$$

where the secondary fields $\phi_i^{(-n)}(z_i)$ are determined by

$$\phi_i^{(-n)}(z_i) = L_{-n}(z_i)\phi_i(z_i) = \oint \frac{dz}{2\pi \mathrm{i}}(z-z_i)^{-n+1}T(z)\phi_i(z_i), \tag{17}$$

where z_i is inside the integration contour. Equivalently, the OPE (16) can be written as

$$T(z)\phi_i(z_i) = \sum_{n\geq 0}(z-z_i)^{n-2}L_{-n}(z_i)\phi_i(z_i). \tag{18}$$

Similarly, the OPE between two stress tensors can be calculated and the result is

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{1}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T(w) + ..., \tag{19}$$

where c is the central charge of the CFT. In principle, OPEs generate descendant fields and provide all the information about the transformations and correlation functions of the primary fields in CFT. The conformal family of a primary field $\phi_i(z)$ with conformal dimension h_i, is defined as an infinite set of descendant fields which are generated by repeated action of L_{-n} operators and it is denoted by

$$[\phi_i(z)] = \{L_{-n_k}L_{-n_{k-1}}...L_{-n_1}\phi_i(z)\}, \tag{20}$$

where $n_k > n_{k-1} > ... > n_1$. We will explain more the action of L_{-n} and concept of conformal family when we discuss the highest weight representations of CFT.

Correlation functions and Ward identity

In this section we concentrate on properties and behaviors of the conformal fields in two-dimensions implied by the conformal symmetry. In fact, a $2d$ CFT is determined by its space of states and its full collection of correlation functions.

We study very briefly the correlation functions of quasi-primary fields and differential equations that the correlation functions of them satisfy. We provide some physical explanations and justifications but for detailed derivations of the results see [BlPl09]. In our language, a correlation function is defined as expectation value of the fields between $SL(2,(C))/\mathbb{Z}_2$-invariant vacuum states $|0>$ on the $2d$ Euclidean space or the complex plane.

The one, two and three point correlation functions are determined by the fact that they are invariant under Möbius transformations $SL(2,\mathbb{C})/\mathbb{Z}_2$ in eq. (7), (whose generators, $L_0, L_{\pm 1}$, annihilates the vacuum). The one-point correlation function vanishes unless $h_i = \bar{h}_i = 0$,

$$< \phi_i(z_i, \bar{z}_i) >= C, \tag{21}$$

where C is independent of z_i and $\bar{z}_i$. The two-point function vanishes unless $h_i = h_j$ and $\bar{h}_i = \bar{h}_j$ and it is given by

$$< \phi_i(z_i, \bar{z}_i)\phi_j(z_j, \bar{z}_j) >= \frac{C_{ij}\delta_{ij}}{(z_i - z_j)^{2h_i}(\bar{z}_i - \bar{z}_j)^{2\bar{h}_i}}, \tag{22}$$

where C_{ij} is the structure constant. For example, the two-point correlation function of stress tensor can be obtained by using the OPE (19) and the fact that the one-point function is zero,

$$< T(z)T(w) >= \frac{c/2}{(z-w)^4}. \tag{23}$$

The three point function of three quasi-primary fields ϕ_1, ϕ_2 and ϕ_3 is fixed up to a constant C_{123} to

$$< \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3) >=$$

$$\begin{aligned} &C_{123}\frac{1}{(z_1-z_2)^{h_1+h_2-h_3}(z_2-z_3)^{h_2+h_3-h_1}(z_1-z_3)^{h_1+h_3-h_2}} \\ &\times\frac{1}{(\bar{z}_1-\bar{z}_2)^{\bar{h}_1+\bar{h}_2-\bar{h}_3}(\bar{z}_2-\bar{z}_3)^{\bar{h}_2+\bar{h}_3-\bar{h}_1}(\bar{z}_1-\bar{z}_3)^{\bar{h}_1+\bar{h}_3-\bar{h}_2}}. \end{aligned} \tag{24}$$

Moreover, the four-point and higher-point correlation functions of quasi-primary fields are restricted partially by Möbius symmetry. The full Virasoro symmetry provides more restrictions on the forms of four and higher point correlation functions. In fact, the Virasoro symmetry generators L_n with $n \leq -2$ do not annihilate the vacuum and they lead to differential equations for correlation functions called Ward identities.

Now, let us study the differential equations that the correlation functions satisfy. An infinitesimal conformal transformation $z \to z + \epsilon(z)$ make a change in the metric and its effect on a correlation function of primary fields is expressed through an integral equation with an insertion of stress tensor which is called *Ward identity*:

$$\begin{aligned} &\oint_C < T(z)\prod_{i=1}^{n}\phi_i(z_i, \bar{z}_i)) > \epsilon(z)dz = \\ &\sum_{i=1}^{n}(h_i\epsilon'(z) + \epsilon(z)(\partial/\partial z_i)) \left\langle \prod_{i=1}^{n}\phi_i(z_i, \bar{z}_i) \right\rangle, \end{aligned} \tag{25}$$

where C is a contour encircling all the points z_j and $\epsilon'(z)$ is the derivative of $\epsilon(z)$. For an arbitrary $\epsilon(z)$ and an arbitrary contour integral, the above equation is valid. This leads to the following differential equation:

$$< T(z)\prod_{i=1}^{n}\phi_i(z_i,\bar{z}_i) >= \sum_{i=1}^{n}[\frac{1}{z-z_i}\partial_{z_i}+\frac{h_i}{(z-z_i)^2}] < \prod_{i=1}^{n}\phi_i(z_i,\bar{z}_i) > . \tag{26}$$

HIGHEST WEIGHT REPRESENTATIONS

Having introduced the operator spectrum of CFT, the notion of quasi-primary and descendant fields, then we need to construct their correspondence states in the Hilbert space. Our aim in this section is to study the representations of the Virasoro algebra and one particular example of that, *highest weight representation* of Virasoro algebra. The idea is that the generators of the Virasoro algebra act on the states of the Hilbert space and either create another states or annihilate them. This is basically analogous to the case of $su(2)$ Lie algebra of spins in quantum mechanics. For simplicity we consider the holomorphic part of the theory.

First, as we discussed let us define the vacuum $|0>$ to be $SL(2,\mathbb{C})$ invariant and it is given by an insertion of the unit operator at the origin of complex plane, $z=0$. Moreover, the action of Virasoro generators on the vacuum is defined such that $L_n|0>=0$ for $n\geq -1$ and $L_{-n}|0>$ for $n\geq 2$ creates the non-trivial states in Hilbert space.

A primary state which is called highest weight state $|h>$ is defined by the action of a primary field operator $\phi(z)$ on the vacuum state as follow:

$$|h>= \lim_{z\to 0}\phi(z)|0>, \tag{27}$$

where $\phi(z)$ is a primary field with conformal dimension h. This is called *field/state correspondence*. Moreover, from the OPE between stress tensor and the primary field we obtain

$$[L_n,\phi(z)] = (z^{n+1}\frac{d}{dz}+(n+1)z^n h)\phi(z), \tag{28}$$

which leads to

$$L_n|h>=0,\ \textit{for}\ n>0, \qquad L_0|h>=h|h> . \tag{29}$$

The action of the central element C on the highest weight state is simply $C|h>=c|h>$.

An arbitrary descendant state $|\vec{n}>$ can be constructed by the action of arbitrary polynomials in $\{L_{-n_i}: n_i\geq 1\}$ on the highest weight state as follow

$$|\vec{n}>= L_{-n_k}L_{-n_{k-1}}...L_{-n_1}|h>, \tag{30}$$

where $n_k>n_{k-1}>...>n_1$. It can be checked that the descendant state $|\vec{n}>$ satisfies the following property:

$$L_0|\vec{n}>= (h+\sum_{i=1}^{k}n_i)|\vec{n}> . \tag{31}$$

The set of the highest weight state and its all descendant states, corresponding to the conformal family $[\phi(z)]$, forms an infinite-dimensional representation, which is completely characterized by its central charge c and the conformal dimension h, and it is called *Verma module*, $V_{c,h}$. The Hilbert space of states is a direct sum of a tensor product of holomorphic and anti-holomorphic Verma modules, over all conformal dimensions of the theory,

$$\bigoplus_{h,\bar{h}} V_{c,h}\otimes\bar{V}_{c,\bar{h}}. \tag{32}$$

The $V_{c,h}$ could have zero-norm and even negative-norm states depending on (c, h). In a non-degenerate unitary CFT these states and their descendants should be removed from the Verma module. The zero-norm state which is called a *null or singular state* is a highest weight which is primary state and a descendant state at the same time. It can be checked that a null state $|\chi>$ at level N satisfies the following equations:

$$L_n|\chi>=0 \quad (for \;\; n>0), \qquad\qquad L_0|\chi>=(h+N)|\chi>, \tag{33}$$

which are the properties that we expect from null state as a highest weight state. For example, in a CFT with the central charge given by $c = \frac{2h}{2h+1}(5-8h)$, a null state at level two is of the following form

$$|\chi>=(L_{-2}-aL_{-1}^2)|h>=0, \tag{34}$$

where $a = \frac{3}{2(2h+1)}$.

The null states in a Verma module generate their own Verma modules V_χ included in the original $V_{c,h}$. Thus, $|\chi>$ and its submodule is orthogonal to the whole Verma module.

A general procedure to determine the null states of the Verma module is through determination of roots of the determinant of a matrix consisting of inner-products between all the basis states of Verma module. Entries of this matrix at level N is of the form $<h|\prod_i L_{k_i}\prod_j L_{-m_j}|h>$, where $\sum_i k_i = \sum_j m_j = N$, for $k_i, m_j \geq 0$. The determinant of a matrix $M_{c,h}^{(N)}$ at level N which is called Kac determinant is given by, [BlPl09],

$$\det M_{c,h}^{(N)} = a_N \prod_{0<p,q\leq N}(h-h_{p,q}(c))^{P(N-pq)}, \tag{35}$$

with

$$h_{p,q}(m) = \frac{((m+1)p-mq)^2-1}{4m(m+1)}, \qquad m = -\frac{1}{2}\pm\frac{1}{2}\sqrt{\frac{25-c}{1-c}}, \tag{36}$$

where a_N is a positive constant and $P(N-pq)$ is a number of partitions of $N-pq$. From a Verma module which contains null states, one can construct an irreducible representation of Virasoro algebra by quotient the null submodules out of the $V_{c,h}$.

It is known that for the following values of the c and h, we have *unitary* representations with $c<1, h\geq 0$,

$$c = 1-\frac{6}{m(m+1)}, \tag{37}$$

for $m = 3, 4, ...$

$$h_{p,q}(m) = \frac{((m+1)p-mq)^2-1}{4m(m+1)}, \tag{38}$$

with $1 \leq p \leq m-1$ and $1 \leq q \leq m$.

The most studied and particular interesting class of conformal field theories is the class of *rational conformal field theories* were first studied by Belavin, Polyakov, and Zamolodchikov in 1984, [BPZ84a] and [BPZ84b]. They are CFTs with the rational conformal weights and central charges that only have finitely many irreducible representations. The simplest example of such RCFTs are the minimal models which exist for certain discrete values of $c<1$. They can be unitary or non-unitary models, specially those minimal models with applications in statistical models are not constrained by unitarity. The first example of minimal model is for $m=3$, $c=1/2$. This minimal model contains unity, energy and spin conformal operators with conformal dimensions $h = 1, 1/2, 1/16$, respectively. This minimal model describes the scaling limit of $2d$ Ising model at criticality. We will discuss more about this at the end of this chapter.

2.3 BOSONIC AND FERMIONIC CFTS

In this short section, we demonstrate two basic examples of conformal field theories which have Lagrangian descriptions, [Ket95]. These two examples are basically, bosonic and fermionic free massless field theory on the plane. Most importantly, they play the central roles in our studies toward two applications in this thesis. We will see that $2d$ free fermion is the crucial object in studies of Ising model and its scaling limit at criticality, on the one hand. On the other hand, *open bosonic string theory* which is an example of $2d$ free bosonic field theory, is the subject of our studies towards understanding the black hole membrane paradigm.

FREE BOSONS

As we mentioned, two-dimensional bosonic CFT is the theory of free massless scalars or bosons on the plane. The central charge of the $2d$ bosonic conformal field theory can be computed by employing the Virasoro algebra and it is $c = 1$. Two important examples of bosonic CFT are *Gaussian free fields* and bosonic strings.

In the Lagrangian formulation, we start by the action of the $2d$ free massless scalar field theory on the complex plane, defined as

$$S = \frac{1}{4\pi} \int d^2 z \partial\Phi(z,\bar{z})\bar{\partial}\Phi(z,\bar{z}), \tag{39}$$

where the integration is on the complex plane. The invariance of the action under conformal transformation (9) requires that the scalar field $\Phi(z,\bar{z})$ has conformal dimensions $(h,\bar{h}) = (0,0)$. The equation of the motion which is obtained by variation of the action with respect to the field is

$$\partial\bar{\partial}\Phi(z,\bar{z}) = 0. \tag{40}$$

The currents, chiral and anti-chiral fields $J(z)$ and $\bar{J}(\bar{z})$, are primary fields and they are defined by

$$J(z) = \partial\Phi(z,\bar{z}), \qquad \bar{J}(\bar{z}) = \bar{\partial}\Phi(z,\bar{z}). \tag{41}$$

The $J(z)$ and $\bar{J}(\bar{z})$ have conformal dimensions $(h,\bar{h}) = (1,0)$ and $(h,\bar{h}) = (0,1)$.

The holomorphic and anti-holomorphic components of the stress tensor are non-zero and they are obtained form the action as follow:

$$T(z) = -\frac{1}{2} : \partial\Phi\partial\Phi : (z) = -\frac{1}{2}[\partial\Phi\partial\Phi(z) + \frac{1}{(z-w)^2}], \qquad \bar{T}(\bar{z}) = -\frac{1}{2} : \bar{\partial}\Phi\bar{\partial}\Phi : (\bar{z}), \tag{42}$$

where :: is the normal order product. In general, normal ordering of any composition of operators in quantum field theory is the rearrangement of the expression such that the annihilation operators are put on the right of the creation operators. In our case, it is a product with subtraction of the divergent parts and can be calculated by means of Wick's formula as $: \partial\Phi\partial\Phi : (z) = \lim_{z\to w}(\partial_z\Phi(z)\partial_w\Phi(w) - \partial_z\partial_w < \Phi(z)\Phi(w) >)$.

The two-point correlation function or propagator of field Φ is the Green function $G(z,\bar{z},w,\bar{w})$, a solution of the $2d$ Laplace equation $\partial_z\partial_{\bar{z}}G(z,\bar{z},w,\bar{w}) = -2\pi\delta^{(2)}(z-w)$,

$$G(z,\bar{z},w,\bar{w}) =< \Phi(z,\bar{z})\Phi(w,\bar{w}) >= -\log|z-w|^2, \tag{43}$$

and consequently correlation functions of currents $J, \bar{J}$ can be obtained as

$$< J(z)J(w) >= -\frac{1}{(z-w)^2}, \quad < \bar{J}(\bar{z})\bar{J}(\bar{w}) >= -\frac{1}{(\bar{z}-\bar{w})^2}, \quad < J(z)\bar{J}(\bar{w}) >= 0. \tag{44}$$

Another interesting class of primary operators are the normal ordered exponentials of free bosons which are called vertex operators; $V_\alpha(z,\bar{z}) =: e^{\mathrm{i}\alpha\Phi(z,\bar{z})}$: with the conformal dimension $(h,\bar{h}) = (\frac{\alpha^2}{2}, \frac{\alpha^2}{2})$. The OPEs of vertex operators can be derived as

$$T(z)V_\alpha(w,\bar{w}) = \frac{\alpha^2/2}{(z-w)^2}V_\alpha(w,\bar{w}) + \frac{1}{z-w}\partial_w V_\alpha(w,\bar{w}) + ...,$$

$$V_\alpha(z,\bar{z})V_\beta(w,\bar{w}) = |z-w|^{2\alpha\beta}V_{\alpha+\beta}(w,\bar{w}) + ...$$

FREE FERMIONS

In this part, two-dimensional free massless fermionic field theory on the complex plane as an example of CFT with central charge $c = \frac{1}{2}$ is introduced very briefly in a general setting and its relation to the scaling limit of the $2d$ Ising model at critical point will be discussed at the end of this part. In fact, we conjecture that the free fermionic fields appear in the scaling limit of the lattice fermions of the Ising model that will be defined in the next chapter. Moreover, formal aspects of the free fermion CFT in bounded domains is briefly summarized in the section (4.2).

Theory of free massless real fermions is defined by its action

$$S = \frac{1}{4\pi}\int d^2z\left(\bar{\psi}(\bar{z})\partial\bar{\psi}(\bar{z}) + \psi(z)\bar{\partial}\psi(z)\right), \tag{45}$$

where ψ is a chiral anticommuting *Majorana-Weyl* fermion and $\bar{\psi}$ is its complex conjugate, [Ket95]. The classical equations of motion are obtained by variation of the action with respects to the fields and they are

$$\begin{aligned} \partial\bar{\psi}(\bar{z}) &= 0, \\ \bar{\partial}\psi(z) &= 0. \end{aligned} \tag{46}$$

The above equations imply that fermions and anti-fermions are holomorphic and anti-holomorphic functions, respectively. Invariance of the action under conformal transformation (9) requires that the ψ and $\bar{\psi}$ are primary fields with conformal dimensions $(h,\bar{h}) = (\frac{1}{2},0)$ and $(h,\bar{h}) = (0,\frac{1}{2})$, respectively. The energy momentum tensor of the fermions is obtained from the action as follow

$$T(z) = -\frac{1}{2} : \psi(z)\partial\psi(z) :, \qquad \bar{T}(\bar{z}) = -\frac{1}{2} : \bar{\psi}(\bar{z})\bar{\partial}\bar{\psi}(\bar{z}) : . \tag{47}$$

The correlation functions of fermions are obtained by means of OPE of fermions as follow:

$$< \psi(z)\psi(w) >= \frac{1}{z-w}, \quad < \bar{\psi}(\bar{z})\bar{\psi}(\bar{w}) >= \frac{1}{\bar{z}-\bar{w}} \quad < \psi(z)\bar{\psi}(\bar{w}) >= 0. \tag{48}$$

All the properties of bosonic and fermionic field theories, such as their Laurent expansion, their OPEs and Ward identities for correlation functions can be obtained simply from the general formulas given in previous sections by putting the appropriate central charges and conformal dimensions of the primary fields. Moreover, the canonical quantization of CFT which is called *radial quantization* can be performed in cylindrical coordinates. It is a standard technique which can be found in any CFT textbook, for example see section (I.2) in [Ket95].

Roughly speaking, the scaling limit of the lattice fermion operators of the Ising model at critical temperature which will be defined in the next chapter gives the free fermion CFT. In bounded domains, this conjecture is verified at the level of correlation functions and differential equations, sections (2.3) and (3.3)-(3.5) in [Za13]. Moreover, as we discussed very briefly, the $c = 1/2$ minimal model explains the $2d$ Ising model at the critical temperature. In fact, the operator content of this model consists of identity, spin and energy operators. The spin operator σ is the continuum limit of the lattice spin operator $\hat{\sigma}_j$ and the energy operator ϵ is the continuum limit of the lattice interaction energy in the Ising model, see "the square lattice Ising model in rectangle" section. In the following, we briefly discuss the relation between the free fermionic sector of the Fock space fields (free fermion CFT) and the primary operators in the setting of the $c = 1/2$ minimal model. In fact, the equivalency between these two theories is at the heart of the exact solutions in $2d$ Ising model.

The energy density operator, $\epsilon(z, \bar{z}) \equiv \varphi^{(\frac{1}{2})}(z, \bar{z})$ with conformal dimension $1/2$ can be written in terms of the fermion and anti-fermion of the free fermionic CFT as the composite of two fermionic fields $\epsilon(z, \bar{z}) = \mathrm{i} : \psi(z)\bar{\psi}(\bar{z}) :$. The correlation functions of energy densities can be easily obtained from the multi-point fermionic correlation functions obtained from the Wick's formula, for example see section (2.3) in [Za13] for the Pfaffian form of the multi-point fermionic correlation function. Two-point correlation function of energy densities can be obtained from the fermionic correlation functions as

$$< \varphi^{(\frac{1}{2})}(z, \bar{z})\varphi^{(\frac{1}{2})}(w, \bar{w}) >= \frac{1}{|z - w|^2}.$$

The relation between spin operator $\sigma(z, \bar{z}) \equiv \varphi^{(\frac{1}{16})}(z, \bar{z})$ with the conformal dimension $1/16$ and the fermion fields is more subtle than the energy case. In fact, one can not get the spin operator from the fermion field in a local way. However, the OPE of the fermion field and the spin operator can be obtained as (see sections (12.2.2) and (12.3.3) [DMS96])

$$\psi(z)\sigma(w, \bar{w}) \sim \frac{1}{(z - w)^{\frac{1}{2}}}\mu(w, \bar{w}), \;\; \bar{\psi}(\bar{z})\mu(w, \bar{w}) \sim \frac{1}{(\bar{z} - \bar{w})^{\frac{1}{2}}}\sigma(w, \bar{w}),$$

where $\mu(w, \bar{w})$ which is called disorder operator is the dual operator to the spin operator with the same conformal dimension and OPE except for the sign;

$$\sigma(z, \bar{z})\sigma(w, \bar{w}) = \frac{1}{|z - w|^{1/4}} + C_{\sigma\sigma\epsilon}|z-w|^{\frac{3}{4}}\epsilon(w, \bar{w}) + ..., \;\; \mu(z, \bar{z})\mu(w, \bar{w}) = \frac{1}{|z - w|^{1/4}} + C_{\mu\mu\epsilon}|z-w|^{\frac{3}{4}}\epsilon(w, \bar{w}) + ...,$$

with $C_{\mu\mu\epsilon} = -C_{\sigma\sigma\epsilon}$. Similarly, the two-point correlation function of spin operators can be obtained as follow

$$< \varphi^{(\frac{1}{16})}(z, \bar{z})\varphi^{(\frac{1}{16})}(w, \bar{w}) >= \frac{1}{|z - w|^{1/4}}.$$

The above two-point correlation functions of energy and spin operators coincide with the continuum limit of the correlation functions of lattice energy and spin operators of the Ising model at the critical temperature, see section (7.4.2) in [DMS96];

$$< \hat{\sigma}_i\hat{\sigma}_{i+n} >= \frac{1}{n^{\frac{1}{4}}}, \;\; < \epsilon_i\epsilon_{i+n} >= \frac{1}{n^2}.$$

Having discussed the basics of CFT, in the following we describe the elements of stochastic Loewner evolution and its relation to CFT.

2.4 STOCHASTIC LOEWNER EVOLUTION

Stochastic (Schramm) Loewner evolution (SLE) was introduced by O. Schramm in 1999, [Sch00], as a way to describe the boundary of percolation clusters and the limit of loop-erased random walk. Roughly speaking, SLE is a family of random non-self-crossing curves in a domain that appear in most $2d$ statistical systems at criticality whose continuum limit respects the conformal invariance. In a physical sense, SLE is an approach towards the description of random curves that appear in the scaling limit of the $2d$ statistical lattice models at critical point. These are curves which appear as the domain walls or interfaces of domains in the scaling limit of the critical two-dimensional statistical systems such as Ising model, percolation etc. It is known for a long time that the scaling limit of these models at criticality exhibits conformal symmetry, therefore we expect that boundaries of domains are conformally invariant curves. Thus, SLE are defined to be conformally invariant curves. The precise meaning of the *conformal invariance* of the SLE curves will be discussed later in this section. Another property of the SLE curves, that is also motivated by the interfaces in statistical mechanics models, is the *domain Markov property* which will be defined carefully in the following. But before studying SLE in details we need to introduce some backgrounds: the theory of conformal mapping and stochastic processes, [KaNi04] and [BaBe06].

CONFORMAL MAPPING THEORY

A *conformal map* $f : D \to D'$ for simply connected domains $D, D' \neq \mathbb{C}$ is a one-to-one map which preserves angles. The heart of the conformal mappings is the *Riemann mapping theorem*. Let $D \subset \mathbb{C}$ be a simply connected domain such that $\mathbb{C} \setminus D$ is not empty, then there is a conformal map from D onto the open unit disk $\mathbb{D}$, $f : D \to \mathbb{D}$. Examples include any bounded domains or domains with well-defined boundary points at infinity such as infinite strip $\mathcal{S} = \{z|0 < \Im z < \pi\}$ and upper-half plane $\mathbb{H} = \{z|\Im z > 0\}$.

In general, the theorem implies that there is a holomorphic conformal mapping $f : D \to D'$ between any two domains D, D'. Thus, any two simply connected domains are conformally equivalent by Riemann mapping theorem. As the result of above theorem and the fact that a map $f : \mathbb{D} \to \mathbb{H}$ is defined by $f(z) = \frac{\mathrm{i}(1+z)}{1-z}$, any simply connected domain can be mapped conformally onto upper half-plane $\mathbb{H}$.

STOCHASTIC PROCESSES

In this part we give a very short summary of some important definitions such as stochastic process, martingales, Brownian motion and stochastic calculus.

A *stochastic process* is a collection of random objects $(X_t)_{t\geq 0}$ indexed by t. In the continuous stochastic processes the index t is interpreted as the continuous time, $t \in [0, \infty)$. Let us recall that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is a collection of sub-sigma algebras satisfying $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$. Sigma algebra $\mathcal{F}_s$ represents the information available at time s. A filtration $(\mathcal{F}_s)_{s\geq 0}$ is all the information generated by the stochastic process X_t up to time s.

A stochastic process $(M_t)_{t\geq 0}$ is called a *martingale* with respect to filtration $(\mathcal{F}_t)_{t\geq 0}$, if it satisfies

$$\mathbb{E}[|M_t|] < \infty, \quad \mathbb{E}[M_t|\mathcal{F}_s] = M_s, \tag{49}$$

for all $0 \leq s \leq t$ and $t \geq 0$. It means that the average value of the process at future, given the information at the present, is equal to the present value of the process.

The Brownian motion plays a key role in the studies about SLE. The standard one-dimensional Brownian motion $(B_t)_{t\geq 0}$ is a continuous stochastic process in time, with $B_0 = 0$ and stationary independent increments $B_t - B_s$ for $t > s \geq 0$. The increment has a Gaussian distribution with mean zero and variance $t - s$. The transition probability density for the Brownian motion going from the position x at time s to position y

at time t is

$$P(y-x,t-s)=\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{(y-x)^2}{2(t-s)}}. \tag{50}$$

Equivalently, the Brownian motion could also be defined as a continuous Gaussian process with

$$\mathbb{E}[B_t]=0, \quad \mathbb{E}[B_tB_s]=min\{t,s\}. \tag{51}$$

STOCHASTIC DIFFERENTIAL EQUATIONS

Let us review a brief intuitive explanation for stochastic calculus which will be used in this section, for a proper treatment of stochastic calculus see [Oks02]. In the stochastic calculus, the differential of time parameter dt should be interpreted as $(\Delta t)_i = t_i - t_{i-1}$ where $0 = t_0 < t_1 < t_2 < ...$ is a discretization of time. Similarly, differentials of Brownian motion dB_t should be interpreted as $(\Delta B)_i = B_{t_i} - B_{t_{i-1}}$. The expressions involving dt or dB_t are always to be integrated. The integrals over dt or dB_t are taken over time intervals that are much longer than the mesh size of time discretization, $(\Delta t)_i$.

Consider a stochastic process $(X_t)_{t\geq 0}$, whose infinitesimal increments have the form

$$dX_t = a(X_t,t)dt + b(X_t,t)dB_t. \tag{52}$$

This is called an stochastic differential equation with the proper mathematical meaning as follow

$$X_s = X_0 + \int_0^s a(X_t,t)dt + \int_0^s b(X_t,t)dB_t,$$

where the integral to be understood as limits of discretizations. Roughly speaking, for any continuously higher-differentiable function f, by using Taylor expansion up to order dt and the fact $(dB_t)^2 = dt$ (intuitively speaking, the size of dB_t is of order $\sqrt{dt}$ because dB_t is a centered Gaussian of variance dt independent of $B|_{[0,t]}$), it can be shown that the stochastic process $f(X_t,t)$ satisfies *Itô formula*:

$$\begin{aligned} df(X_t,t) = \quad & [a(X_t,t)f'(X_t,t) + \dot{f}(X_t,t) + \frac{1}{2}b(X_t,t)^2 f''(X_t,t)]dt \\ & + b(X_t,t)f'(X_t,t)dB_t, \end{aligned} \tag{53}$$

where $f'(X_t,t)$ and $f''(X_t,t)$ are first and second order derivatives of $f(X_t,t)$ with respect to X and $\dot{f}(X_t,t)$ is the derivative of $f(X_t,t)$ with respect to t. The above formula can be easily generalized to a multi-dimensional case.

It can be shown that, if the first term in the Itô formula which is called the drift term vanishes, then $f(X_t,t)$ is a local martingale.

SLE CURVES

Generally speaking, SLE is a stochastic conformally invariant process in fictitious time, that generates a conformally invariant family of non-self-crossing random curves, [La05]. The SLE curves are defined by solutions of a differential equation, called Loewner equation with random input. The meaning of this sentence will become clear as we proceed.

Let us consider a simply connected domain D with arbitrary number of marked boundary points and also arbitrary number of marked interior points. Then, different SLE curves are defined depending on the starting and ending of the curve on these boundary and interior points. There are two major different types of SLE curves: i) *chordal* SLE with two boundary points in a domain; the chordal SLE curve starts at a boundary point and ends at another boundary point, ii) *radial* SLE with a boundary and an interior points in a domain; the radial SLE curve starts at a boundary point and end at an interior point. Moreover, there are also some other variants of the SLE, such as multiple SLE with many boundary points etc. In the following presentation of the SLE, its properties and relations to CFT, we restrict ourselves to the simplest fundamental case of chordal SLE i.e. SLEs defined for simply connected domains and depend on two marked points on the boundary of the domain, denoted by x_0 and x_∞.

In order to define the SLE curves, which are in fact nothing but the probability measures on random curves, we start by writing down two assumptions that are motivated from the scaling limit of statistical lattice models at criticality. Moreover, we have to assume first that the curves exist and they are given by the Loewner chain, a collection of conformal maps that will be defined in the following. Consider the simplest case, chordal SLE, with probability measures $\{\mu_D(x_0, x_\infty)\}$, indexed by simply connected domain D and distinct boundary points $x_0, x_\infty \in \partial D$. Then we assume that these measures have two properties: *Conformal invariance* and *domain Markov property*.

• **Conformal Invariance:** let us define a conformal map: $f : D \to D'$. It maps all the points in the domain D to some other points in domain D'. It also maps the boundary points to $f(x_0)$ and $f(x_\infty)$. As we mentioned, the SLE is defined to be a stochastic conformally invariant process. It means that the probability measure associated to $(D; x_0, x_\infty)$ which we denote it by $\mu_D(x_0, x_\infty)$, and the probability measure associated to $(f(D); f(x_0), f(x_\infty))$, $\tilde{\mu}_{f(D)}(f(x_0), f(x_\infty))$, are related via a push-forward relation; $\tilde{\mu} = f * \mu$. In other words, if a random curve γ in D that connect two boundary points $a, b \in \partial D$ has the law $\mu_D(a, b)$, then its image $f \circ \gamma$ has the law $\mu_{f(D)}(f(a), f(b))$. Therefore, it is sufficient to define the SLE in a simply connected reference domain like upper half-plane $\mathbb{H}$ and then the definition of the SLE in any other simply connected domain is given by the conformal mapping. In our case, the case of chordal SLE, it is sufficient to consider the SLE in $(\mathbb{H}, 0, \infty)$.

• **Domain Markov property:** for any given initial segment $\gamma[0, s]$ of the chordal SLE curve in domain $(D; x_0, x_\infty)$, the conditional distribution (law) of the remaining part of the curve, $\mu_D(x_0, x_\infty)|_{\gamma[0,s]}$, is the same as the original distribution on the remaining domain, $\mu_{D\setminus\gamma[0,s]}(\gamma(s), x_\infty)$, i.e. the law of chordal SLE in $(D \setminus \gamma[0, s]; \gamma(s), x_\infty)$.

The SLE curves can be studied from a complex analysis point of view. In order to describe the SLE curves, we consider the complementary of the curves and the conformal mapping uniformizing them.

In general, simple curves $\gamma(t) : [0, \infty] \to \mathbb{C}$ growing in the upper half plane $\mathbb{H}$ can be described by the behavior of their shrinking complementary $H_t := \mathbb{H} \setminus \gamma[0, t]$. Therefore, we use the Loewner equation that describes the evolution of the conformal uniformizing mappings, $g_t : H_t \to \mathbb{H}$, from the complement of the curves, $H_t := \mathbb{H} \setminus \gamma[0, t]$, to $\mathbb{H}$. There is a unique map g_t such that satisfies the *hydrodynamic normalization*, $\lim_{z\to\infty}(g_t(z) - z) = 0$. Then the expansion of g_t at infinity with a choice of time parametrization is given by

$$g_t(z) = z + \frac{2t}{z} + O(1/z^2). \tag{54}$$

Loewner has proved that the maps g_t satisfy a differential equation

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - X_t}, \qquad g_0(z) = z, \tag{55}$$

where $X : [0, \infty) \to \mathbb{R}$ is a continuous function with $X_0 = 0$. The function X_t is in fact the image of the tip of curve under the map g_t, $X_t = g_t(\gamma(t))$. This differential equation is called the Loewner equation.

Following the above construction, the chordal SLE_κ, indexed by a parameter $\kappa > 0$, can be described in (D, x_0, x_∞). Schramm observed that the above two assumptions on probability measures translate into assumptions on the driving function in the Loewner equation; $X_t = \sqrt{\kappa}B_t$ where B_t is a one-dimensional standard Brownian motion. The proof of the fact that the only driving force for the random curves that are satisfying the conformal invariance, Markov property and the reflection symmetry (in the case of half-plane) is the one-dimensional Brownian motion can be performed. A simple physical argument is based on the conformal invariance and Markov properties which lead to the following statement: all the increments $\Delta_n = X_{(n+1)\delta t} - X_{n\delta t}$ of the process X_t are independent identically distributed random variables for $\delta t > 0$. The only process that satisfies this is Brownian motion with a possible drift term, which the reflection symmetry set the drift term to zero. For further details see theorem (3.1) in [Car05].

The above two assumptions (conformal invariance and domain Markov property) lead to the *Schramm principle*. The Schramm principle classifies all possible conformal invariant random curves which can be described by Loewner evolution and satisfy domain Markov property. In the case of chordal SLE, it states that the chordal SLE curves are the only non-self crossing curve processes in (D, x_0, x_∞) whose measure are conformally invariant and satisfy domain Markov property and they are characterized uniquely by one parameter, $\kappa \geq 0$. In other words, there can be no other conformally invariant chordal random curves with domain Markov property except the ones whose half plane Loewner chain (g_t) has driving process $(\sqrt{\kappa}B_t)_{t\geq 0}$ for some $\kappa \geq 0$.

One can argue in the opposite direction and use the Schramm observation to define the chordal SLE. Therefore, the chordal SLE_κ in $(\mathbb{H}, 0, \infty)$ can be defined as a collection of conformal maps g_t of the form (54) obtained by solving the Loewner equation (55) for all $z \in \mathbb{H}$ with random driving force, $X_t = \sqrt{\kappa}B_t$,

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \qquad g_0(z) = z. \tag{56}$$

Then, it has been proved in [RoSc05] that chordal SLE_κ is generated by a curve and also one can prove that the chordal SLE curve that is defined in this way has the above two properties: conformal invariance and domain Markov property. Therefore, chordal SLE_κ is a conformally invariant law on random curves in D from x_0 to x_∞ and in the half plane it is obtained by solving the Loewner equation driven by the one-dimensional standard Brownian motion B_t.

A curve $\gamma(t)$ is called *chordal SLE_κ trace* in $(\mathbb{H}, 0, \infty)$ and its defined by $\gamma(t) = \lim_{\epsilon\to 0} g_t^{-1}(X_t + \mathrm{i}\epsilon)$. The SLE trace continuously depends on time, $t \geq 0$, and $\lim_{t\to\infty} \gamma(t) = \infty$. The existence of the limit and its continuous dependence on $t \geq 0$ is proved in [RoSc05]. Moreover, it has been proved in [RoSc05] that the trace of the chordal SLE_κ in $(\mathbb{H}, 0, \infty)$ is transient; $\lim_{t\to\infty} \gamma(t) = \infty$.

The characteristics of geometry of the SLE curves are dependent of the parameter κ and it has been proved that the chordal SLE_κ have three phases, [La05], for physical justification of the following result see section (3.4.1) in [Car05] and section (4.2) in [KaNi04]:

- **For $\kappa \leq 4$, γ is a simple non-intersecting curve;**
- **For $4 < \kappa < 8$, γ is a self-intersecting curve;**
- **For $\kappa \geq 8$, γ is a space filling curve.**

It has been proved in [Bef02], [Bef08] and [La10] that the *Hausdorff (fractal) dimension* of the SLE_κ trace, $dim[\gamma(0,\infty)]$, for $0 \leq \kappa \leq 8$, is $1 + \frac{\kappa}{8}$ and for $\kappa \geq 8$ is 2.

The proof of the above results are beyond the scope of this thesis and for the logical completeness of the thesis we just summarized some very important results and properties of the SLE. The main feature of the theory that is proved in some cases is that the SLE curve with one parameter κ, chordal SLE_κ, appears in the two-dimensional statistical lattice models as the rigorous scaling limit of the interfaces of the model. Some important examples are SLE_6 in percolation, SLE_4 in Gaussian free fields and SLE_3 in Ising model.

Let us just recall some of the applications and properties of the SLE. I) boundary hitting probabilities: the probabilities that the SLE curve intersects some specific sequence of intervals on the real axis and ii) the SLE can be used to calculate Cardy's crossing probability formula for critical percolation: that is the probability that a percolation cluster connects the left side of a rectangle to the right side of that.

2.5 CFT/SLE CORRESPONDENCE

Recently, deep connections between conformal field theory and Schramm Loewner evolution have been found, for review of different aspects of these relations see [BaBe06], [Car05] and [Gr06]. These relations led to a new active interdisciplinary subject between physics and mathematics with lots of collaborations between physicists and mathematicians. The CFT/SLE relations shed a new light on both topics and have many advantages from different points of view. On the one hand, SLE provides a rigorous approach to statistical mechanics of the planar lattice models which was not available via the path integral approach. On the other hand, calculations and predictions that CFT provide in SLE-related areas are difficult or impossible to obtain by present techniques of SLE.

The connection between CFT and SLE is based on the fact that the boundary point where the interface emerges can be viewed as the insertion of a boundary condition changing operator. This operator is a Virasoro primary field ψ degenerate at level two i.e. it has a vanishing descendant field at level two. We will see that it is related to the martingale property of SLE observables. In order for CFT/SLE correspondence to happen, SLE parameter κ and central charge c and scaling dimension h of the field ψ in CFT moduli should satisfy

$$c_\kappa = \frac{(3\kappa - 8)(6-\kappa)}{2\kappa}, \qquad h = \frac{6-\kappa}{2\kappa}. \tag{57}$$

In other words, SLE_κ appears in interfaces of the CFT's with Virasoro central charge $c_\kappa < 1$ which is invariant under duality $\kappa \leftrightarrow 16/\kappa$. Before explaining these deep relations more carefully with some calculations, we have to introduce the notion of the CFT in bounded domains, boundary conformal field theory.

BOUNDARY CONFORMAL FIELD THEORY

CFTs in domains with boundaries are of great interest and importance because of many applications such as boundary critical behaviors of lattice models and SLE in mathematics.

In this chapter we have reviewed the two-dimensional CFT in the whole complex plane. However, it is possible to study CFT in a domain with boundaries such as half-plane $\mathbb{H}$ or any arbitrary domain D. The starting point of this study is the pioneering works of Cardy, [Car84] and [Car89] which lead to the important topic of *boundary conformal field theory* (BCFT). The simplest example of BCFT is the CFT in the upper-half plane $\mathbb{H}$. Once we understand CFT, its OPE, correlation functions etc. in the simple case of half plane then the results can be transformed to any arbitrary simply connected domain D which is conformally equivalent to $\mathbb{H}$, by using the Riemann mapping theorem. Having discussed the preliminary definitions and results in CFT which are also valid on the half plane, we try to define BCFT in the following way.

The boundary conformal field theory consists of boundary conformal fields located on the boundary of the domain, ∂D, as well as bulk fields inside the domain D. We denote a field by $\phi_i(x_i)$ with $x_i \in D$. The bulk and boundary fields are either primary or secondary fields and the primary ones transform under the conformal active domain transformations between domains according to their conformal weights, h_i. As an example, let us consider the transformation rules for a primary field ϕ_i and quasi-primary field, stress tensor T, on the domain D to the half-plane $\mathbb{H}$. Under a conformal domain transformation, $g : D \to \mathbb{H}$, we have

$$\begin{aligned} \phi_i(x_i) &= g'(x_i)^{h_i} \phi_i(g(x_i)), \\ T(x_i) &= g'(x_i)^2 T(g(x_i)) + \frac{c}{12} S_g(x_i), \end{aligned} \tag{58}$$

where c is the central charge of CFT, $g'(x_i)$ is the derivative of g with respect to x_i and S_g is the Schwarzian derivative of g. Therefore, the correlation functions of a product of primary fields transform homogeneously under the conformal mapping $f : D \to D'$ from domain D to domain D' as follow

$$\left\langle \prod_{i=1}^{n} \phi_i(x_i) \right\rangle_D = \prod_{i=1}^{n} |f'(x_i)|^{h_i} \left\langle \prod_{i=1}^{n} \phi_i(f(x_i)) \right\rangle_{D'}. \tag{59}$$

In this chapter, we have introduced the Virasoro generators and their relation to the notion of descendant fields. Recall that, if the central charge $c = \frac{2h(5-8h)}{(2h+1)}$, then a null field at level two can be constructed from the primary field ψ, with a conformal dimension h, as follow

$$L_{-2}\psi - aL_{-1}^2\psi, \tag{60}$$

where $a = \frac{3}{2(2h+1)}$. The primary field ψ is called degenerate at level two if the above expression vanishes. This plays an important role in CFT/SLE correspondence. In fact, the field ψ, a primary and degenerate at level two, is a boundary condition changing (B.C.C.) operator and it is inserted in a point where the SLE curve starts. The B.C.C. operator is an operator which change the boundary conditions on the right and left sides of its insertion point. For example, as we will see in the next chapter, in the case of Ising model the sign of spin flips on the left and right sides of the insertion point of the fermion operator.

The next important ingredient of CFT for the CFT/SLE correspondence is the null field differential equation. From the expression (60), the null field equation for a primary field ψ degenerate at level two and the equation $L_{-1}\psi(z) = \partial\psi(z)$ imply that $L_{-2}\psi(z) = \psi^{(-2)}(z) = a\partial_z^2\psi(z)$. Now, we can use the OPE (16) to write

$$L_{-2}\psi(z) = \lim_{w\to z}\left[T(w)\psi(z) - \frac{h\psi(z)}{(w-z)^2} - \frac{\partial_z\psi(z)}{w-z}\right]. \tag{61}$$

Finally, replacing this into the Ward identity (26), the differential equation for half-plane correlation functions of primary fields $\phi_i(z_i)$ with conformal dimension h_i and a Virasoro field $\psi(z)$ degenerate at level two is obtained as follow

$$\left(\frac{3}{2(2h+1)}\partial_z^2 + \sum_{i=1}^{n}[\frac{1}{z_i - z}\partial_{z_i} - \frac{h_i}{(z_i-z)^2}]\right) < \psi(z)\prod_{i=1}^{n}\phi_i(z_i) >_{\mathbb{H}} = 0, \tag{62}$$

Having briefly introduced BCFT, we will continue the explanation of CFT/SLE correspondence.

We will elaborate on the CFT/SLE correspondence in two closely related directions. First, we give a group theoretical operator formalism description for this correspondence and second, we describe a more physical and intuitive approach based on statistical mechanics and field theory. The latter one is basically the relation between CFT null field differential equation for correlation functions and martingale property of SLE observables.

GROUP THEORETICAL OPERATOR FORMALISM

The idea of this section is to present the CFT/SLE correspondence by showing the relation between the group theoretical formulation of the SLE and the operator formalism of CFT. The goal is to clarify the relation between SLE and representation of the Virasoro algebra, [BaBe06] and [BaBe03]. Roughly speaking, we want to study the SLE as a formal stochastic process in the group of conformal transformations, through the first order stochastic differential equation (SDE) on Lie group. In order to be more clear we will consider the simplest case, the chordal SLE.

STATE OF THE SLE_κ CURVE

We start with a short review on the conformal stochastic processes. Consider a stochastic motion on a Riemann surface Σ. A stochastic flow $f_t(z)$ in a coordinate system for some open subset of Σ, $z \in U \subset \mathbb{C}$, satisfies an equation of motion, $df_t = dt\sigma(f_t(z)) + d\xi_t\rho(f_t(z))$ where ξ_t is a Brownian motion with covariance $\mathbb{E}[\xi_t\xi_s] = \kappa\min(t,s)$. By using the map $\phi : U \to V$, the equation of motion for $f_t^\phi = \phi\circ f_t\circ\phi^{-1}$ can be written in a different coordinate system V, with the help of $It\hat{o}$ formula,

$$df_t^\phi = dt(\sigma^\phi \circ f_t^\phi) + d\xi_t(\rho^\phi \circ f_t^\phi), \tag{63}$$

where $\sigma^\phi \circ \phi = \phi'\sigma + \frac{\kappa}{2}\phi''\rho^2$ and $\rho^\phi \circ \phi = \phi'\rho$. As we know, the paths of the particles on manifolds are related to vector fields. One can see that, eq. (63) expresses that the holomorphic vector fields of this equation are $w_{-1} \equiv \rho(z)\partial_z$ and $w_{-2} \equiv \frac{1}{2}(-\sigma(z) + \frac{\kappa}{2}\rho(z)\rho'(z))\partial_z$. The two vector fields generate a Lie algebra and this Lie algebra will give us a group theoretical formulation of the problem. In this sense, we want to associate to the flow f_t an element of a group. We assume that there exists such a group and we call it N. Furthermore, we assume that there is a linear space O of holomorphic functions F, that the group acts faithfully on this space and the elements of the group $\mathfrak{g}_{f_t} \in N$ act by composition $\mathfrak{g}_{f_t} \cdot F = F \circ f_t$. Thus we associate an element $\mathfrak{g}_{f_t} \in N$ to each f_t. In this sense, f_t can be interpreted as a random process $\mathfrak{g}_{f_t}$ on N.

Then $It\hat{o}$ formula shows that

$$\mathfrak{g}_{f_t}^{-1} \cdot d\mathfrak{g}_{f_t} = (dt\sigma + d\xi_t\rho)F' + dt(\frac{\kappa}{2})\rho^2 F''), \tag{64}$$

or equivalently

$$\mathfrak{g}_{f_t}^{-1} \cdot d\mathfrak{g}_{f_t} = dt(-2w_{-2} + \frac{\kappa}{2}w_{-1}^2) + d\xi_t w_{-1}. \tag{65}$$

We will see that by using the appropriate linear space O and the group N for the SLE processes this equation determines the relation between SLE and CFT in the operator formalism. In fact, w_{-2} and w_{-1} generate a Lie algebra which can be embedded in the Virasoro algebra, a central extension of the Witt algebra.

Similar to the above arguments, in the following first we introduce a function f_t for the SLE process and then we define a corresponding group element of conformal transformation of f_t whose germ at infinity is given. The group of this conformal transformations can be identified with a Lie group generated by negative Virasoro generators. Elements of this Virasoro group which we denote by G_f satisfy a SDE which defines a stochastic Markov process on this group.

In the upper half plane for the chordal SLE, we define a conformal map $f_t(z) = g_t(z) - \xi_t$, where g_t is the Loewner map for SLE from 0 to ∞ and ξ_t is the multiple of Brownian motion. It satisfies a differential equation $df_t = \frac{2dt}{f_t} - d\xi_t$. Let us define a group N_- of germs of holomorphic functions at infinity of the from $z + \sum_{m\leq -1} f_m z^{m+1}$. Similar to the above discussion, to any f_t of the latter form we associate an element $\mathfrak{g}_{f_t} \in N_-$, then by using $It\hat{o}$ formula, we have

$$\mathfrak{g}_{f_t}^{-1} d\mathfrak{g}_{f_t} = dt(\frac{2}{z}\partial_z + \frac{\kappa}{2}\partial_z^2) - d\xi_t\partial_z = dt(-2l_{-2} + \frac{\kappa}{2}l_{-1}^2) + d\xi_t l_{-1}, \tag{66}$$

where we have used $w_{-n} = l_{-n}$ in eq. (65) and recall that $l_n = -z^{n+1}\frac{\partial}{\partial z}$ are generators of Witt algebra.

In the CFT, the operator l_n are promoted to Virasoro generators L_n which satisfy commutation relation (8). The group of germs of conformal transformations $z \to z + \sum_{m\leq -1} f_m z^{m+1}$ at infinity can be identified with the Lie group Vir_- obtained from the exponentiating the generators L_n (n<0). More precisely, to any function $f \in z + \mathbb{C}[[z^{-1}]]$ of the form $f(z) = z + \sum_{m\leq -1} f_m z^{1+m}$, one can associate an operator $G_f \in \overline{\mathcal{U}(\mathfrak{vir}_-)} = \prod_{d=0}^{\infty}\mathcal{U}(\mathfrak{vir}_-)_d$ which is a completion of the universal enveloping algebra $\mathcal{U}(\mathfrak{vir}_-) = \bigoplus_{d=0}^{\infty}\mathcal{U}(\mathfrak{vir}_-)_d$ of the Virasoro subalgebra $\mathfrak{vir}_-$ generated by $L_n(n < 0)$. The representations of the Virasoro algebra are not automatically the representation of N_-, but the highest weight representation of the Virasoro algebra can be extended in such a way that the N_- can be embedded in the appropriate completion $\overline{\mathcal{U}(\mathfrak{vir}_-)}$ of the enveloping universal algebra of the Virasoro subalgebra $\mathfrak{vir}_-$ generated by L_n (n<0), [BaBe06] and [BaBe03]. Therefore, we can associate to $\mathfrak{g}_{f_t} \in N_-$ an operator $G_{f_t} \in \overline{\mathcal{U}(\mathfrak{vir}_-)}$ in CFT which acts on the appropriate representation of Virasoro algebra so that $\mathfrak{g}_{f_t} \to G_{f_t}$ is a homomorphism. The operator G_{f_t} as an operator implementing the conformal map $f_t(z)$ in CFT Hilbert space acts by conjugation,

$$G_{f_t}^{-1}\psi_h(z)G_{f_t} = |f_t'(z)|^h\psi_h(f_t(z)), \tag{67}$$

where $\psi_h(z)$ is a primary field with conformal dimension h. Using the homomorphism $\mathfrak{g}_{f_t} \to G_{f_t}$, the stochastic equation for stochastic operator G_{f_t} in CFT follows from eq. (66),

$$G_{f_t}^{-1}dG_{f_t} = dt(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2) + d\xi_t L_{-1}, \;\; G_{t=0} = 1, \tag{68}$$

where L_n are generators of conformal transformations in CFT.

If the state $|\psi>$ is a highest weight vector of the Virasoro algebra representation and degenerate at level two with the central charge and conformal dimension in eq. (57), then it satisfies $(-2L_{-2}+\frac{\kappa}{2}L_{-1}^2)|\psi>=0$. Then the transformed state in the domain H_t with $f_t : H_t \to \mathbb{H}$ is given by $G_{f_t}|\psi>$. The $It\hat{o}$ formula for the transformed state can be obtained by the action of both sides of eq. (68) on the state $|\psi>$ as follow,

$$dG_{f_t}|\psi>= G_{f_t}L_{-1}|\psi> d\xi_t. \tag{69}$$

Since the drift term in the above equation is zero, we observe that the $G_{f_t}|\psi>$ is a local martingale. Therefore, the scalar product $<\phi|G_{f_t}|\psi>$ for any state $<\phi|$ is a martingale and it satisfies a stochastic version of the conservation law. In fact, for $t \geq s$ we have

$$\mathbb{E}[<\phi|G_{f_t}|\psi> |\{G_{f_u}\}_{u\leq s}] =<\phi|G_{f_s}|\psi> . \tag{70}$$

Since G_{f_t} is the intertwining operator between domain $\mathbb{H}$ and random domain H_t, then the correlation function of an arbitrary operator $\mathcal{O}$ in domain H_t, can be constructed from the transformed states, $G_{f_t}|\psi>$, as follow

$$<\mathcal{O}>_{H_t}=<\psi|G_{f_t}^{-1}\mathcal{O}G_{f_t}\psi(0)|0>=<\psi|\mathcal{O}G_{f_t}|\psi>, \tag{71}$$

where we used $\psi(0)|0>=|\psi>$ and $<\psi|G_{f_t}^{-1}=<\psi|$. Since, $G_{f_t}|\psi>$ is a local martingale, then $<\mathcal{O}>_{H_t}$ is also a local martingale observable, and therefore we have

$$\mathbb{E}[<\mathcal{O}>_{H_t}] =<\mathcal{O}>_{\mathbb{H}} . \tag{72}$$

It shows that the CFT correlation functions in domain H_t are in average time independent.

Let us review our observation. The state $|\psi>$ of the SLE_κ at time $t=0$ in $(\mathbb{H},0,\infty)$ which keeps the track of boundary conditions is defined formally by the action of a primary field on the vacuum, $|\psi>=\lim_{z\to 0}\psi(z)|0>$. It can be easily seen that the translated state $|\psi_x>=\psi(x)|0>$ satisfies $|\psi_x>=e^{xL_{-1}}|\psi>$ where L_{-1} is infinitesimal generator of translations. The state of the SLE in the CFT Fock space at time t in $(\mathbb{H}\setminus\gamma[0,t],\gamma(t),\infty)$, is then given by the action of an operator G_{f_t}, implementing the conformal transformation on the space of states and it is $G_{f_t}|\psi>$. This state is called a generating function of local martingales.

STATISTICAL MARTINGALE PROPERTY OF CORRELATION FUNCTIONS

In the following we present the martingale property in the context of spin systems in statistical physics. Consider a correlation function of the operator $\mathcal{O}(z_1,...,z_n)$ as a product of general local fields that depend on spin configurations with $z_i \in D$ and two marked boundary points $a,b\in\partial D$. The normalized correlation functions can be written in the statistical mechanics language as follow

$$<\mathcal{O}(z_1,...,z_n)>_{D;a,b}=\frac{\sum_{\sigma\in\mathcal{C}_{D;a,b}}e^{-\beta H(\sigma)}\mathcal{O}(z_1,...,z_n)}{Z_{D;a,b}}, \tag{73}$$

where the sum is over all the spin configurations $\sigma\in\mathcal{C}_{D;a,b}$ in domain D with marked boundary points a,b, the function $H(\sigma)$ is the Hamiltonian and the partition function is defined as $Z_{D;a,b}=\sum_{\sigma\in\mathcal{C}_{D;a,b}}e^{-\beta H(\sigma)}$. For simplicity, we will denote the partition function by Z_D.

Furthermore, we can assume that there is a path γ_t starting at point a in domain D and its tip point x is in the bulk of the D until it will reach the boundary point b at time $t=T$. Notice that t is a fictitious parameterizing time. Then, we can write the correlation function as follow

$$<\mathcal{O}(z_1,...,z_n)>_{D;a,b}=\frac{\sum_{\gamma_t}\sum_{\sigma\in\mathcal{C}_{D;a,b},\gamma^{(\sigma)}=\gamma_t}e^{-\beta H(\sigma)}\mathcal{O}(z_1,...,z_n)}{Z_D}, \tag{74}$$

where the first sum is over all possible shape of paths denoted by curves γ_t and the second sum is over spin configurations which are compatible with any specific curve γ_t. It can be shown that the correlation function can be written in the following form

$$< \mathcal{O}(z_1, ..., z_n) >_{D;a,b}= \sum_{\gamma_t} \mathbb{P}\left[\gamma[0,t] = \gamma_t\right] < \mathcal{O}(z_1, ..., z_n) >_{D\setminus\gamma_t;x,b}, \tag{75}$$

where the probability on the paths is defined as

$$\mathbb{P}\left[\gamma[0,t] = \gamma_t\right] = \frac{\sum_{\sigma\in\mathcal{C}_{D;a,b},\gamma^\sigma=\gamma_t} e^{-\beta H(\sigma)}}{Z_D} = \frac{Z_{D\setminus\gamma_t}}{Z_D}, \tag{76}$$

and the correlation function on domain $D \setminus \gamma_t$ is defined as

$$< \mathcal{O}(z_1, ..., z_n) >_{D\setminus\gamma_t;x,b}= \frac{1}{Z_{D\setminus\gamma_t}} \sum_{\sigma\in\mathcal{C}_{D\setminus\gamma_t}} e^{-\beta H(\sigma)}\mathcal{O}(z_1, ..., z_n). \tag{77}$$

Therefore, the relation between correlation functions on different domains is obtained as

$$< \mathcal{O}(z_1, ..., z_n) >_{D;a,b}= \sum_{\gamma_t} \frac{Z_{D\setminus\gamma_t}}{Z_D}\mathcal{O}(z_1, ..., z_n)_{D\setminus\gamma_t,x,b}. \tag{78}$$

This leads to the domain Markov property of correlation functions

$$\begin{aligned} \mathbb{E}_{D;a,b}[\mathcal{O}(z_1, ..., z_n)|\gamma[0,t] = \gamma_t] &= \mathbb{E}_{D\setminus\gamma_t;x,b}[\mathcal{O}(z_1, ..., z_n)], & (79)\\ < \mathcal{O}(z_1, ..., z_n) >_{D;a,b} |_{\gamma[0,t]=\gamma_t} &= < \mathcal{O}(z_1, ..., z_n) >_{D\setminus\gamma_t;x,b} . & (80) \end{aligned}$$

Let us rewrite $\mathbb{E}_{D;a,b}[\mathcal{O}(z_1, ..., z_n)|\gamma[0,t] = \gamma_t]$ as $\mathbb{E}[\mathcal{O}(z_1, ..., z_n)|\mathcal{F}_t] \equiv< \mathcal{O} >_t$. Then, by construction $< \mathcal{O} >_t$ is a martingale with respect to $\mathcal{F}_t$,

$$\mathbb{E}[< \mathcal{O} >_t |\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[\mathcal{O}(z_1, ..., z_n)|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[\mathcal{O}(z_1, ..., z_n)|\mathcal{F}_s] =< \mathcal{O} >_s$$

which we have used $\mathcal{F}_s \subset \mathcal{F}_t$ for $t > s$.

CFT correlation functions and SLE martingale observables

One of the most important aspects of the relations between CFT and SLE is the relation between correlation functions of CFT and SLE martingale observables. Moreover, there are SLE observables or some functions of them which have the martingale property. It means that the drift term in the It$\hat{o}$ formula of the SLE observables, vanishes. Therefore, we can naively formulate this corresponding as the following: A large collection of observables of the SLE on domain D can be written in terms of the correlation functions of primary fields including the B.C.C field on the domain D which the corresponding CFT correlation functions on $\mathbb{H}$ satisfy the null field differential equations.

The first important example is the relation between correlation functions of primary fields including Virasoro field $\psi(x)$, with conformal dimension h, degenerate at level two with null descendant, and SLE martingales in the chordal case. In this section we study the relations between null field differential equation and vanishing of the drift term in the It$\hat{o}$ formula for the chordal SLE on $\mathbb{H}$. In the case of chordal SLE, the correlation function of number of primary fields $\Psi_i(Y_i)$ with conformal dimension h_i, $\mathcal{O} = \prod_i^N \Psi_i(Y_i)$ on the domain H_t where the curve is cut from the half plane is obtained by the insertion of two boundary fields, one at infinity and the other one at the tip of the curve, γ_t. The boundary field at infinity creates the state

$< \psi| =< 0|\psi(\infty)$ and furthermore, the boundary fields are primary fields degenerate at level two. Then, the normalized correlation function on domain H_t is written as

$$< \mathcal{O} >_{H_t} = \frac{< \psi(\infty)\mathcal{O}\psi(\gamma_t) >_{H_t}}{< \psi(\infty)\psi(\gamma_t) >_{H_t}}. \tag{81}$$

By using the conformal map $g_t : H_t \to \mathbb{H}$ the normalized correlation function can be written as

$$< \mathcal{O} >_{H_t} = \frac{< \psi(\infty)\mathcal{O}_t\psi(X_t) >_{\mathbb{H}}}{< \psi(\infty)\psi(X_t) >_{\mathbb{H}}}, \tag{82}$$

where the $\mathcal{O}_t = \prod_{i=1}^{N}(g_t'(Y_i)^{h_i}\Psi_i(Y_i))$ is the image of $\mathcal{O}$ under the conformal map g_t and $X_t = g_t(\gamma_t) = \sqrt{\kappa}B_t$ in the case of chordal SLE. Moreover, since the normalization factor is trivial the correlation function can be written explicitly as CFT correlation function

$$< \mathcal{O} >_{H_t} = \prod_{i=1}^{N} g_t'(Y_i)^{h_i} < \psi| \prod_{i=1}^{N} \Psi_i(g_t(Y_i))\psi(X_t)|0 >= J_t Z. \tag{83}$$

where the Jacobian is $J_t = \prod_{i=1}^{N} g_t'(Y_i)^{h_i}$ and by using the simpler notation $x = X_t$, $y_i = g_t(Y_i)$, Z can be written as $Z =< \Psi_1(y_1)...\Psi_N(y_N)\psi(x) >_{\mathbb{H}}$. The claim is that the correlation function is a local martingale observable of chordal SLE. The reason is that for the above SLE observable the drift term in the It$\hat{o}$ derivative vanishes. To show that, first we need the null field differential equation for the Z which is the following

$$\left[\frac{3}{2(2h+1)}\frac{\partial^2}{\partial x^2} - \sum_{i=1}^{N}\left(\frac{h_i}{(x-y_i)^2} + \frac{1}{x-y_i}\frac{\partial}{\partial y_i}\right)\right] Z = 0. \tag{84}$$

Using the It$\hat{o}$ formula for the $\psi(x)$, $d\psi(x) = \psi'(x)dx + \frac{3}{2}\psi''(x)dt$, and Loewner equation for $g_t(z)$ and its derivative with respect to z, one can get the It$\hat{o}$ derivative of $< \mathcal{O} >_{H_t}$ as follow

$$d < \mathcal{O} >_{H_t} = J_t \left[dX_t\partial_x + dt\left(\frac{3}{2(2h+1)}\frac{\partial^2}{\partial x^2} - \sum_{i=1}^{N}\left(\frac{h_i}{(x-y_i)^2} + \frac{1}{x-y_i}\frac{\partial}{\partial y_i}\right)\right)\right] Z. \tag{85}$$

Then, from eqs. (84) and (85), since the drift term vanishes, we observe that the SLE observable is a local martingale.

There are specific realizations of CFT/SLE correspondence, as explained above, for different CFTs and SLEs. One important example of this correspondence is the relation between Gaussian free fields and SLE_4, [SchSh09] and [KaMa11]. Especially, the Coulomb gas formalism and its relation to SLE has been studied extensively, [Gr06]. In the section (4.2), we explore another realization of CFT/SLE, namely the relation between free fermion fields and SLE_3 in the scaling limit of Ising model at critical temperature.

Finally, it is good to mention that there is an alternative point of view for CFT/SLE correspondence based on path integral formulation found by J. Cardy, [Car05]. However, in this thesis we focus on the introduced approach to CFT/SLE correspondence in [BaBe06].

3 PLANAR ISING MODEL AND DISCRETE HOLOMORPHICITY

In this chapter, first we review some standard topics in the $2d$ Ising model such as transfer matrix formalism, free fermions etc. and then we will briefly explain the basic definitions and standard results in the subject of discrete holomorphicity and its relation to planar Ising model.

3.1 PLANAR ISING MODEL, TRANSFER MATRIX, FREE FERMIONS

In this section we explain first the transfer matrix formalism which is an approach towards an exact solution of two-dimensional Ising model on a rectangle with specific boundary conditions. Second, the Fock space representation in Ising model is briefly reviewed.

Before starting to explain the transfer matrix formalism and Fock representation of the Ising model we try to give a brief description of the other exact solutions of the Ising model. In the introduction we briefly reviewed the history of exact methods in the Ising model. However, the complete review of all the exact approaches to the Ising model is beyond the scope of this thesis.

In addition to the Onsager-Kauffman solution which is basically the diagonalization of the transfer matrix, there are two important approaches to the exact solution of the Ising model. The first one is based on the Jordan-Wigner transformation that builds a set of anticommuting fermionic creation and annihilation operators from spin operator and the Bogoliubov diagonalization of the transfer matrix. This approach exhibits the fermionic character of the Ising model from a different perspective. The fermionic technique is underlying most of the studies about the exact results in Ising model, [SML64]. In fact, the conjecture is that the fermions of the Jordan-Wigner transformation in the continuum limit are identified with the free fermion fields introduced in the previous part.

The other approach is the construction of the disorder operator μ in [KaCe71]. In recent studies, the spin and disorder operators are used to define the fermions on the lattice [RaCa07]. However, the focus of this thesis is not on this approach and we keep to the original definition of the lattice fermions as it is explained in [Pal07].

3.1.1 FOCK SPACE FORMALISM IN ISING MODEL

Analogy between quantum field theories and scaling limit of lattice statistical systems near criticality has been studied extensively. However, operator formalism of QFT can be applied even for lattice models outside of criticality. We explicitly investigate one of the fundamental examples, fermionic techniques in two-dimensional Ising model.

The fermionic structure of the Ising model was used also in studies of conformal field theory of the critical Ising model, [DMS96]. However, the transfer matrix formalism is mainly suitable for the rigorous study of the model in the full plane or enough symmetric domains, whereas the formulations of conformal invariance of the model should be performed in domains of arbitrary shapes.

THE SQUARE LATTICE ISING MODEL IN RECTANGLE

We start with a brief review on basic definitions about two-dimensional Ising model [Pal07]. Ising model is the simplest nontrivial lattice model consisting of spins $\{\sigma_i\}$ on the vertices of the lattice and interacting by nearest neighborhood interactions. In contrary to one-dimensional model, as shown by Peierls [Pe36], Onsager [Ons44], Kramers and Wannier [KrWa41] and Kaufman and Onsager [KaOn49], two-dimensional Ising model possess a second order phase transition at critical inverse temperature ($\beta = \frac{1}{k_B T}$), $\beta_c = \frac{1}{2} \ln(\sqrt{2} + 1)$.

Consider a rectangular square lattice $\mathbb{Z}^2$ in a finite domain $\Lambda_{M,N}$,

$$\Lambda_{M,N} = \{(j,i) \in \mathbb{Z}^2 \mid |j| \leq M, |i| \leq N\}. \tag{86}$$

The Hamiltonian or the energy of a configuration $\sigma \in \mathcal{C}_\Lambda = \{\pm 1\}^\Lambda$, which is a spin configuration in the set of all configurations in the domain Λ composed of all possible arrangements of up and down spins on the lattice sites, is defined by

$$E_\Lambda(\sigma) = -\sum_{<x,y>\subset\Lambda} J\sigma_x\sigma_y, \tag{87}$$

where J is strength of the interaction between spins in both horizontal and vertical directions and $< x, y >$ denotes the pairs of sites that are nearest neighbors. For the rest of this chapter we assume isotropic and homogenous coupling and for simplicity we put $J = 1$. Gibbs measure for the probability of the configurations is

$$P_T(\sigma) = \frac{1}{\mathcal{Z}_\Lambda(T)} \exp\left(-\beta E_\Lambda(\sigma)\right), \tag{88}$$

where the partition function of the model is

$$\mathcal{Z}_\Lambda(T) = \sum_{\sigma\in\mathcal{C}_\Lambda} \exp\left(-\beta E_\Lambda(\sigma)\right). \tag{89}$$

The expected value of the product of spin variables, $\sigma_A = \prod_{i\in A}\sigma_i$ in a subset A of the domain Λ, a finite collection of sites, defines an observable of the system by

$$< \sigma_A >_\Lambda = \frac{1}{\mathcal{Z}_\Lambda} \sum_{\sigma\in\mathcal{C}_\Lambda} \sigma_A \exp\left(-\beta E_\Lambda(\sigma)\right). \tag{90}$$

In presence of boundary conditions, the sums over all configurations in partition function and correlation functions are restricted to the configurations that satisfy the boundary conditions.

THE TRANSFER MATRIX FORMALISM

As we mentioned, operator formalism in quantum field theory is one of the main approaches to study systems with finite and infinite degrees of freedom such as lattice models and their scaling limit. Transfer matrix method is an analogues powerful technique in analysis of the one and two-dimensional Ising model and other related statistical models. This method provide an approach to an exact solution of the two-dimensional Ising model. For the first time, Kramers and Wannier applied the transfer matrix method in the two-dimensional Ising model, [KrWa41]. The transfer matrix is a linear operator which the partition function and free energy can be read off of a matrix element of some high power of it, see eq. (97). Also correlation functions of operators can be expressed using the transfer matrix and the operators acting on the same vector space, see eq. (98).

In this section we review the machinery of the transfer matrix formalism for the square lattice Ising model. The aim of this part is to obtain the observables of the Ising model by the transfer matrix formalism. In the transfer matrix formalism, the sums over all configurations in partition function or correlation functions are divided into the multiple sums over the configurations of the rows, $\mathcal{C}_\Lambda(\text{row}) = \{\pm 1\}^{2M+1}$. The row configurations are indexed by "$i = -N, -N+1, ..., N-1, N$", and the columns in each row configuration are indexed by following set of integers

$$I_M = \{-M, -M+1, ..., M\}. \tag{91}$$

In order to explicitly calculate the partition function and correlation functions, it is convenient to introduce a Hilbert space of dimension 2^{2M+1} and the basis which are indexed by $\mathcal{C}_\Lambda$(row)

$$\mathcal{H} = \bigotimes_{j=-M}^{M} \mathbb{C}_j^2 = \mathbb{C}_{-M}^2 \otimes ... \otimes \mathbb{C}_M^2. \tag{92}$$

The map between elements of $\mathcal{C}_\Lambda$(row) and basis of the vector space is

$$\mathcal{C}_\Lambda(row) \ni \sigma \to e_\sigma = \bigotimes_{j=-M}^{M} \begin{bmatrix} \frac{1+\sigma_j}{2} \\ \frac{1-\sigma_j}{2} \end{bmatrix}. \tag{93}$$

Spin operator acting on this vector space has the following representation:

$$\hat{\sigma}_j = 1 \otimes ... \otimes 1 \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes 1 \otimes ... \otimes 1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_j. \tag{94}$$

Suppose that σ and $\rho \in \mathcal{C}_\Lambda$(row), then we define symmetrized transfer matrix $V_M : \mathcal{C}_\Lambda(\text{row}) \to \mathcal{C}_\Lambda(\text{row})$ by $V_M = V_1^{\frac{1}{2}} V_2 V_1^{\frac{1}{2}}$ and V_1 is a diagonal matrix

$$(V_1)_{\rho\sigma} = \begin{cases} \exp\left(\sum_{j=-M}^{M-1} \beta\sigma_j\sigma_{j+1}\right) & \text{if } \sigma \equiv \rho \\ 0 & \text{otherwise,} \end{cases} \tag{95}$$

and the matrix elements of V_2 are given by

$$(V_2)_{\rho\sigma} = \begin{cases} \exp\left(\sum_{j=-M}^{M} \beta\rho_j\sigma_j\right) & \text{if } \sigma_{-M} = \rho_{-M} \text{ and } \sigma_M = \rho_M \\ 0 & \text{otherwise,} \end{cases} \tag{96}$$

where ρ_j and σ_j are the row configurations evaluated as $j-th$ column.

As a final remark of this section, we translate the expression for the partition function (up to a normalization) and correlation functions in presence of fixed plus or minus boundary conditions into the vector space representation as follow:

$$\mathcal{Z}_\Lambda =< e_\sigma^N | V_1^{\frac{1}{2}} V_M^{2N} V_1^{\frac{1}{2}} | e_\sigma^{-N} >, \tag{97}$$

and

$$< \sigma_A >_\Lambda = \frac{< e_\sigma^N | V_1^{\frac{1}{2}} V_M \sigma_{A_{N-1}} V_M \sigma_{A_{N-2}} ... \sigma_{A_{-N+1}} V_M V_1^{\frac{1}{2}} | e_\sigma^{-N} >}{\mathcal{Z}_\Lambda}, \tag{98}$$

where e_σ^N and e_σ^{-N} are Hilbert space representation of the N^{th} and $-N^{th}$ row configurations and they play the role of boundary states, and σ_{A_i} denotes the restriction of σ_A to the $i-$th row. In the case of the partition function and correlation functions of spin operators, the boundary conditions of the lattice is fixed to either plus or minus boundary conditions on all four sites of the lattice.

CLIFFORD ALGEBRA AND CLIFFORD GROUP

As we see in this section, exact solvability of the two-dimensional Ising model leads to an observation that the transfer matrix can be written as exponential of a quadratic expression in Clifford algebra generators eqs. (108) and (109), and as a representation of the Clifford algebra, the space on which the transfer matrix acts is a fermionic Fock space. In this sense the two-dimensional Ising model is a fermionic field theory. The field theory is free in the sense that the transfer matrix and its spectrum are simply expressible in terms of its action on the one-particle sector of the fermionic Fock space only. In this part we demonstrate a representation of the Clifford algebra and group in the case of the Ising model. First, we give some general definitions in Clifford algebra and then we introduce a representation of Clifford algebra in Ising model.

Assume that W is a finite-dimensional complex vector space with a nondegenerate complex bilinear form denoted by $(\cdot,\cdot)$. In general, a Clifford algebra $Cliff(W)$ on the vector space W is defined as an associative algebra with unit e which is generated by elements in W satisfying the following relation

$$ab + ba = (a, b)e. \tag{99}$$

The basis of $Cliff(W)$ consists of e and the monomials $w_{k_1}w_{k_2}...w_{k_n}$ for $1 \leq k_1 < k_2 < ... < k_n \leq dim(W)$ where w_k are basis of W. Therefore, the dimension of $Cliff(W)$ is $2^{dim(W)}$.

In the case of Ising model, the Clifford algebra generators $\{\frac{p_k}{\sqrt{2}}, \frac{q_k}{\sqrt{2}}\}$ as they will be defined below are orthonormal basis of a complex vector space W'_M which is a finite-dimensional complex Hilbert space

$$\begin{aligned} W'_M &= Span(\{p_k|k \in I_M - \frac{1}{2}\} \cup \{q_k|k \in I_M + \frac{1}{2}\}) \\ &= W_M \oplus (\mathbb{C}_{p_{-M-\frac{1}{2}}} + \mathbb{C}_{q_{M+\frac{1}{2}}}). \end{aligned} \tag{100}$$

Elements of the Clifford algebra, p_k and q_k are indexed by half-integers k. We define a finite-dimensional, irreducible spin representation of Clifford algebra $Cliff(W'_M)$, the so called Brauer-Weyl representation, acting on $\bigotimes_{j=-M}^{M} \mathbb{C}_j^2$ space. For $k \in I_M - \frac{1}{2}$, we define

$$p_k = \left\{ \prod_{j=-M}^{k-\frac{1}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_j \right\} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_{k+\frac{1}{2}}, \tag{101}$$

and for $k \in I_M + \frac{1}{2}$:

$$q_k = \left\{ \prod_{j=-M}^{k-\frac{3}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_j \right\} \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}_{k-\frac{1}{2}}. \tag{102}$$

The elements p_k and q_k form an irreducible representation of the Clifford algebra as it can be easily checked that they satisfy the anti-commutation Clifford relations:

$$\begin{aligned} p_k p_l + p_l p_k &= 2\delta_{kl}, \\ q_k q_l + q_l q_k &= 2\delta_{kl}, \\ p_k q_l + q_l p_k &= 0. \end{aligned} \tag{103}$$

Any vector $v \in W'_M$ can be expanded as a complex linear combination of basis vectors, p_k and q_k,

$$v = y_0(v)\frac{p_{-M-\frac{1}{2}}}{\sqrt{2}} + x_0(v)\frac{q_{M+\frac{1}{2}}}{\sqrt{2}} + \sum_{k=-M+\frac{1}{2}}^{M-\frac{1}{2}} x_k(v)\frac{q_k}{\sqrt{2}} + y_k(v)\frac{p_k}{\sqrt{2}}, \tag{104}$$

and the conjugated vector is defined by

$$\bar{v} = \overline{y_0(v)}\frac{p_{-M-\frac{1}{2}}}{\sqrt{2}} + \overline{x_0(v)}\frac{q_{M+\frac{1}{2}}}{\sqrt{2}} + \sum_{k=-M+\frac{1}{2}}^{M-\frac{1}{2}} \overline{x_k(v)}\frac{q_k}{\sqrt{2}} + \overline{y_k(v)}\frac{p_k}{\sqrt{2}}.$$

Moreover, the Hermitian symmetric inner product between two vectors v and w is defined by

$$< v, w >= \overline{x_0(v)}x_0(w) + \overline{y_0(v)}y_0(w) + \sum_{k=-M+\frac{1}{2}}^{k=M-\frac{1}{2}} \overline{x_k(v)}x_k(w) + \overline{y_k(v)}y_k(w), \tag{105}$$

and furthermore the bilinear form is defined by $(v, w) =< \bar{v}, w >$.

Another crucial concept in this part is *induced rotation*. Assume V is a linear transformation on $\bigotimes_{j=-M}^{M} \mathbb{C}_j^2$ and it is invertible. The induced rotation of V is defined as $T(V) : W'_M \to W'_M$ such that for all $v \in W'_M$ we have

$$T(V)v = VvV^{-1}. \tag{106}$$

The induced rotation $T(V)$ determines V up to an scalar factor. If there exists $T(V)$, V is called an element of the Clifford group. Both spin operators and the transfer matrix are elements of the Clifford group. In the case of transfer matrix, the induced rotation preserves the bilinear form and it is a complex orthogonal map. To calculate the action of the induced rotation of the transfer matrix $T(V_M)$ on vector v in the above equation, we use the spin representation, in which the transfer matrix is defined as $V_M = V_1^{\frac{1}{2}} V_2 V_1^{\frac{1}{2}}$, and by using the following identities

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_j = ip_{j-\frac{1}{2}}q_{j+\frac{1}{2}}, \ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_j \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_{j+1} = iq_{j+\frac{1}{2}}p_{j+\frac{1}{2}}, \tag{107}$$

V_1 and V_2 can be obtained in spin representation as

$$V_1 = \exp\left(\mathrm{i}\beta \sum_{k=-M+\frac{1}{2}}^{M-\frac{1}{2}} q_k p_k \right), \tag{108}$$

$$V_2 = (2\sinh(2\beta))^{M-\frac{1}{2}} \exp\left(\mathrm{i}\beta^* \sum_{j=-M+\frac{1}{2}}^{M-\frac{3}{2}} p_k q_{k+1} \right). \tag{109}$$

For derivation of the above results see [Pal07] (section 1.2.1) or [HKZ12] (proposition 8). As we see, the transfer matrix is given by the exponentials of the Clifford algebra generators and therefore it is an element of the Clifford group. These are crucial results which are necessary to obtain the induced rotation of the p and q and finally the fermion operator of the Ising model.

FOCK REPRESENTATION

In this part we give a short presentation of the Fock representation of the Clifford algebra which are the basis of important results in this thesis. The complete exposition of the Fock space formalism is developed in sections (3.3)-(3.5) in [HKZ12].

We are interested in the representation of the Clifford algebra which arises in an isotropic splitting of W. In general, an isotropic splitting of $W = W_+ + W_-$ is a direct sum decomposition in which the subspaces, W_+ and W_-, are isotropic subspaces that means they have the property that the bilinear form on them is identically zero. In the case of Ising model, we are interested in an isotropic splitting of W which is called Hermitian polarization. This polarization is defined such that W_+ and W_- are orthogonal with respect to the Hermitian inner product and its written as $W = W_+ \bigoplus W_-$. To each polarization of the latter form there is a Fock representation of the Clifford algebra on W which lives on the alternating tensor algebra $Alt(W_+)$ defined by

$$Alt(W_+) = \mathbb{C} \oplus Alt^1(W_+) \oplus Alt^2(W_+) \oplus ... \oplus Alt^n(W_+),$$

where $Alt^j(W_+)$ is the space of alternating j tensors over W_+ and $n = dim(W_+)$; $Alt^j(W_+) \ni v = v_1 \wedge v_2 \wedge ... \wedge v_j = \frac{1}{j!}\sum_{\sigma\in S_j} Sgn(\sigma)v_{\sigma(1)} \otimes v_{\sigma(2)} ... \otimes v_{\sigma(j)}$ for $v_i \in W_+, i = 1, ..., j$ and S_j denotes the permutation group of j letters. Due to the nondegeneracy of the bilinear form, two subspaces W_+ and W_- are dual to each other. Let $(x^\dagger_\alpha)^n_{\alpha=1}$ be a basis of W_+ and $(x_\alpha)^n_{\alpha=1}$ be a dual basis of W_-. The Fock representation associated to the Hermitian polarization is given by the action of creation operator $x^\dagger_\alpha \in W_+$ and annihilation operator $x_\alpha \in W_-$ which are defined by

$$x^\dagger_\alpha.(x^\dagger_{\beta_1} \wedge ... \wedge x^\dagger_{\beta_p}) = x^\dagger_\alpha \wedge x^\dagger_{\beta_1} \wedge ... \wedge x^\dagger_{\beta_p} \in Alt^{p+1}(W_+),$$

$$x_\alpha.(x^\dagger_{\beta_1} \wedge ... \wedge x^\dagger_{\beta_p}) = \sum_{i=1}^{p}(-1)^{i-1}(x_\alpha, x^\dagger_{\beta_i})x^\dagger_{\beta_1} \wedge ... \wedge x^\dagger_{\beta_{i-1}} \wedge x^\dagger_{\beta_{i+1}} \wedge ... \wedge x^\dagger_{\beta_p} \in Alt^{p-1}(W_+),$$

where $(x_\alpha, x^\dagger_{\beta_i}) = \delta_{\alpha,\beta_i}$.

The vacuum vector of the Fock representation is defined as a unique vector that is annihilated by annihilation operators in W_-. The vacuum vector in $Alt(W_+)$ is usually denoted by $|0 >= 1 \oplus 0 \oplus ... \oplus 0$.

In a physical polarization, the vacuum state is an eigenvector associated to the largest eigenvalue of the transfer matrix V_M and the isotropic splitting which the Fock representation is associated to that can be defined as $W_M = W^+_M \bigoplus W^-_M$ such that W^+_M is spanned by eigenvectors of T_M (restriction of $T(V_M)$ on W_M) with eigenvalues less than one and W^-_M is spanned by eigenvectors with eigenvalues bigger than one. The spectral analysis of the T_M (section (3.5) in [HKZ12]) shows that: 1) T_M does not have eigenvalue 1. 2) $W_M = W^+_M \bigoplus W^-_M$ is an orthogonal direct sum decomposition. 3) Since T_M is an complex orthogonal mapping, for any $v, w \in W^+_M$ eigenvectors of T_M, $T_M v = av$ and $T_M w = bw$ with eigenvalues $a, b < 1$, we have $(v, w) = (T_M v, T_M w) = ab(v, w)$. This leads to $(v, w) = 0$, which means that W^+_M is an isotropic subspace. The similar argument shows that W^-_M is an isotropic subspace.

FERMION OPERATORS AND DIRAC EQUATION

In general, free fermions play an important role in many different areas of physics and mathematical physics. In our context, it is believed that "the two-dimensional Ising model is a free fermion". As we discussed, the Fock representation lives on the alternating tensor algebra. Therefore, Fock space approach to the two-dimensional Ising model, using the alternating tensor algebra, provides a natural framework for describing the antisymmetric statistics of many particle states for free fermions.

First, we define lattice fermion and anti-fermion operators on the mid-points of horizontal edges of two-dimensional rectangular lattice. In spin representation we define

$$\psi_k = A_\psi(q_k + p_k), \tag{110}$$
$$\bar{\psi}_k = A_{\bar{\psi}}(p_k - q_k), \tag{111}$$

where $A_\psi = \frac{\lambda^{-3}}{\sqrt{2}}$ and $A_{\bar{\psi}} = \overline{A_\psi}$ for $\lambda = e^{\frac{i\pi}{4}}$. Moreover, one can simply check that the fermions and anti-fermions satisfy

$$\psi_k\psi_l + \psi_l\psi_k = 4A^2_\psi\delta_{kl}, \;\; \bar{\psi}_k\bar{\psi}_l + \bar{\psi}_l\bar{\psi}_k = 4A^2_{\bar{\psi}}\delta_{kl}, \;\; \psi_k\bar{\psi}_l + \bar{\psi}_l\psi_k = 0.$$

Roughly speaking, we expect that fermions satisfy the Dirac equation. However, derivation of the Dirac equation is not a result of this thesis but in order to clarify the discussion, we demonstrate the appropriate form of the Dirac equation for Ising fermions without derivations. The Dirac equation is obtained from the action of induced rotation of the transfer matrix on generators of Clifford algebra and fermion operators in a complicated way. It has been shown in [Pal07] (sections 1.3 and 4.2) that in a naive scaling limit the fermion $\Psi(x)$ which is defined by

$$\Psi(x) = \begin{bmatrix} \psi(x) \\ \bar{\psi}(x) \end{bmatrix} = \begin{bmatrix} A_{\psi}(q(x) + p(x)) \\ A_{\bar{\psi}}(-q(x) + p(x)) \end{bmatrix}, \tag{112}$$

satisfies the massive Dirac equation with mass $\mu(\beta) = \frac{\beta_c - \beta}{\beta}$,

$$\mathcal{D}\Psi(x) - \mu(\beta)\Psi(x) = 0, \tag{113}$$

where

$$\mathcal{D} = \begin{bmatrix} 0 & 2\partial \\ 2\bar{\partial} & 0 \end{bmatrix}, \tag{114}$$

and the definition of partial complex derivatives are

$$\partial = \frac{1}{2}(\frac{\partial}{\partial x_1} - \mathrm{i}\frac{\partial}{\partial x_2}) \quad \text{and} \quad \bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial x_2}). \tag{115}$$

The basic idea is that the massless free fermions which satisfy the Dirac equation in the limit $\mu \to 0$,

$$\bar{\partial}\psi(x) = 0, \tag{116}$$

$$\partial\bar{\psi}(x) = 0, \tag{117}$$

describe the scaling limit of Ising model at critical temperature. Moreover, we observe that, free fermions (anti-fermions) are holomorphic (anti-holomorphic) functions.

BRANCHING OF THE FERMION AT SPIN INSERTION

Another important property of a fermion is its branching around the spin operator insertion. The induced rotation of the spin operator for ψ_k and $\bar{\psi}_k$ can be easily obtained from induced rotation of the spin operator for the p_k and q_k as follow: the induced rotations of the spin operator can be obtained easily as

$$s_j q_k = sgn(j - k) q_k, \tag{118}$$

$$s_j p_k = sgn(j - k) p_k, \tag{119}$$

where $s_j v = \hat{\sigma}_j v \hat{\sigma}_j^{-1}$ for any vector $v \in W'_M$. Therefore, we obtain

$$s_j \psi_k = -sgn(k - j)\psi_k \quad \textit{and} \quad s_j \bar{\psi}_k = -sgn(k - j)\bar{\psi}_k. \tag{120}$$

We observe that fermion operator at point k (anti-)commute with spin operator inserted on the (left) right hand side of k.

In the next section we introduce the graphical techniques to obtain the partition function and correlation functions in Ising model.

3.1.2 THE HIGH AND LOW TEMPERATURE EXPANSIONS AND DUALITY

In statistical physics, the method of cluster expansions provides a systematic way of computing partition functions and correlation functions as power series. Cluster expansion has variety of applications to lattice models such as Ising model.

The Ising model can be studied by series expansions, so called low-temperature and high-temperature expansions. General definitions of low-temperature and high-temperature expansions of partition function and correlation functions are based on the graphical expansions of the model which consist of collection of spin configurations.

In this part, we will present the low-temperature and high-temperature expansions of the partition functions, then we establish the relation between the low-temperature and high-temperature expansions, using the concept of *duality*.

The partition function of the $2d$ Ising model is given by the following expression:

$$\mathcal{Z}(\beta) = \sum_{\sigma \in \{\pm 1\}^V} \prod_{<x,y>} \exp{(\beta \sigma_x \sigma_y)}, \tag{121}$$

where the sum is over all spin configurations, $\sigma \in \{\pm 1\}^V$, which V is the collection of vertices of the square lattice.

In low-temperature expansion, a spin configuration is represented by the set of its interfaces and in fact there is a bijection between spin configurations with plus boundary conditions and contours of graphical low-temperature expansion. This correspondence can be obtained by setting the rule that spin values on the dual lattice change sign when traversing an edge and otherwise remains the same. In the low-temperature regime by using the expansion of the product in partition function (121) in power series of $\frac{1}{\beta^*}$, $Z(\beta^*)$ in the low temperature can be obtained as

$$\mathcal{Z}_{low}(\beta^*) = e^{(\beta^* E)} \sum_{\sigma \in \mathcal{C}_L} e^{-2\beta^* L(\sigma)}, \tag{122}$$

where the sum is over collections of dual edges in all configurations $\sigma \in \mathcal{C}_L$, consisting of loops, surrounding plus (or minus) spin clusters in the minus (or plus) spins backgrounds, E is the total number of dual edges of lattice and $L(\sigma)$ is the number of dual edges of loops in each configuration σ.

In the high-temperature regime although there is no bijection between spin configurations and contour representations (boundaries on the dual lattice), the partition function can be represented in an expansion called high-temperature expansion. By using the identity

$$e^{(\beta \sigma_x \sigma_y)} = \cosh(\beta) + \sigma_x \sigma_y \sinh(\beta) = \cosh \beta (1 + \sigma_x \sigma_y \tanh \beta), \tag{123}$$

and the expansion of the product over edges in partition function as power series of β, after some simplifications the partition function for the Ising model on square lattice with free boundary conditions in high-temperature regime can be obtained as the following:

$$\mathcal{Z}_{high}(\beta) = 2^V (\cosh(\beta))^E \sum_{\sigma \in \mathcal{C}_H} [\tanh(\beta)]^{L(\sigma)}, \tag{124}$$

where the sum is over collections of edges in all configurations $\sigma \in \mathcal{C}_H$, consisting of closed loops on the lattice, V is the number of vertices, E is the total number of edges of lattice and $L(\sigma)$ is the number of edges of loops in each configuration σ. Indeed, low and high temperature expansions hold in any temperature regime, since they are exact expansions on the lattice.

As it is mentioned, there is a duality between high and low-temperature expansions of the Ising model on the square lattice, called Kramers-Wannier duality, [KrWa41]. In fact, duality transformation is a duality between ordered and disordered phases of the model.

The high-low temperature duality transformation can be summarized briefly as following:

1) there is a mapping or a duality relation between high and low-temperature phases of the Ising model:

$$\tanh(\beta) \leftrightarrow e^{-2\beta^*}, \tag{125}$$

where β^* is dual inverse temperature. In fact the low-temperature and high-temperature expansions mapped to each other through the identification $\tanh(\beta) = e^{-2\beta^*}$,

2) by using trigonometric identities and the above identification it is easy to show that the dual inverse temperature β^*, is related to β, by the following equation:

$$\sinh(2\beta)\sinh(2\beta^*) = 1, \tag{126}$$

and

3) at critical temperature or self-dual point $\beta = \beta^* = \beta_c$, the equation $\tanh(\beta_c) = e^{-2\beta_c}$ imply

$$\beta_c = \frac{1}{2}\ln(\sqrt{2}+1) = 0.440686... \tag{127}$$

In the following we give an introduction to discrete holomorphicity, especially the notion of s-holomorphicity and other related subjects.

3.2 Discrete holomorphicity

In this part we introduce discrete holomorphicity and its special realization, the s-holomorphicity. We will also discuss about its role in the $2d$ Ising model, [DuSm11]. The linear equations between lattice fermions were found for the Ising model in 80's, [DoPo88]. At the end of this section, the parafermionic observables will be discussed for the critical $2d$ Ising model. Moreover, discrete holomorphicity has been applied to other models such as $O(N)-$models etc. [RaCa07], [IkCa09] and [Car09].

Analyticity and holomorphicity have been provided a mathematical toolbox to study the statistical systems including continuum models as well as lattice models. However, in lattice models, we need a discrete version of analyticity and holomorphicity. A trivial example of discrete holomorphicity is the discrete version of the Cauchy-Riemann equation on a two-dimensional lattice.

Assume, G is a simply connected subgraph of the square lattice $\mathbb{Z}^2$ consisting of entire plaquette. A discrete holomorphic observable $F(z_{ij})$ is a complex-valued function defined at the midpoints z_{ij} of edges (ij) of G and it satisfies the discrete version of the contour integral,

$$\sum_{(ij)\in\mathcal{F}} F(z_{ij})(z_i - z_j) = 0, \tag{128}$$

which the sum is over the edges of that face $\mathcal{F}$ of G. Equivalently, F is discrete analytic if it satisfies the discrete Cauchy-Riemann equation, $\mathrm{i}\partial_x F = \partial_y F$. The discrete version of this on the lattice in fig. (1) is

$$F(N) - F(S) = \mathrm{i}\left(F(E) - F(W)\right). \tag{129}$$

Notice that eqs. (128) and (129) are both relating values of the function at four distinct points via an equation. Roughly speaking, we have one complex equation and two complex unknowns per plaquette. By Dirichlet boundary conditions on the boundary of the lattice, one can observe that these equations are not enough generically to determine the bulk points of the lattice in terms of the boundary points. In the following, we will define an stronger notion of discrete holomorphicity known as s-holomorphicity.

3.2.1 S-holomorphicity

The definition of s-holomorphicity

S-holomorphicity is a notion of discrete holomorphicity for functions defined at the midpoints of edges of square lattice domains. It admits generalizations to isoradial graphs with slightly different formulation from

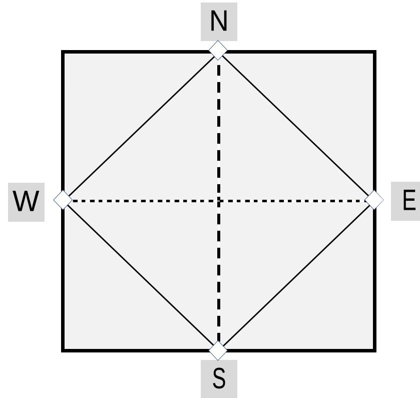

Figure 1: Discrete holomorphicity

the definition below, [ChSm11]. However, s-holomorphicity is a stronger condition than discrete Cauchy-Riemann equation, in the sense that s-holomorphicity implies discrete Cauchy-Riemann equation. For detailed study of discrete complex analysis with s-holomorphic functions look at [ChSm11].

Consider a square lattice domain, a rectangle $D = \Lambda_{M,N}$, a subgraph of $\mathbb{Z}^2$ consisting of entire plaquettes. The set of midpoints of edges of D is denoted by $D_\diamond$. To each pair of adjacent midpoints we associate a line in the complex plane. Associated lines to the edges of the medial graph in the complex plane are,

$$l_{NE} = \omega\mathbb{R}, \quad l_{NW} = \omega^{-1}\mathbb{R}, \quad l_{SE} = \omega^3\mathbb{R}, \quad l_{SW} = \omega^{-3}\mathbb{R}, \tag{130}$$

with $\omega = e^{\frac{\mathrm{i}\pi}{8}}$.

A function $F : D_\diamond \to \mathbb{C}$ is called *s-holomorphic* if for all pairs of adjacent midpoints the orthogonal projections of F (at the two points) to the corresponding line coincide. For $u \in \mathbb{C}$ of unit modulus, $|u| = 1$, the projection to the line $u\mathbb{R}$ of a complex number ζ is $\frac{1}{2}(\zeta + u^2\bar{\zeta})$. Then, for $N, W, S, E \in D_\diamond$, the four midpoints of edges around a plaquette, a s-holomorphic function F satisfies

$$\begin{aligned} F(N) + \lambda\overline{F(N)} &= F(E) + \lambda\overline{F(E)} \\ F(N) + \lambda^{-1}\overline{F(N)} &= F(W) + \lambda^{-1}\overline{F(W)} \\ F(S) + \lambda^{3}\overline{F(S)} &= F(E) + \lambda^{3}\overline{F(E)} \\ F(S) + \lambda^{-3}\overline{F(S)} &= F(W) + \lambda^{-3}\overline{F(W)}, \end{aligned} \tag{131}$$

where $\lambda = \frac{1+\mathrm{i}}{\sqrt{2}}$. The standard discretizations of the Cauchy-Riemann equations can be derived from these equations. Therefore, s-holomorphicity is naturally a notion of discrete holomorphicity.

The s-holomorphicity is defined in a general way, independent of any lattice models. Although, we will see that, fermions of Ising model at critical temperature satisfy the s-holomorphicity relations. However, in order to define massive s-holomorphicity, we restrict ourselves to the case of the Ising model. Outside the critical point of the Ising model, there are functions solving the modification of the s-holomorphicity equation, known as massive s-holomorphicity. A massive holomorphic function $F\colon D_\diamond \to \mathbb{C}$ satisfies

$$\begin{aligned} \tau F(N) - \overline{F(N)} &= \lambda^3 F(E) + \tau\lambda\overline{F(E)} \\ F(N) - \tau\overline{F(N)} &= \tau\lambda^{-3} F(W) + \lambda^{-1}\overline{F(W)} \\ F(S) + \tau\overline{F(S)} &= \tau\lambda^{-3} F(E) + \lambda^{3}\overline{F(E)} \\ \tau F(S) + \overline{F(S)} &= \lambda^{3} F(W) + \tau\lambda^{-3}\overline{F(W)}, \end{aligned} \tag{132}$$

where $\tau = \frac{\alpha+\mathrm{i}}{\alpha-i} = \frac{-1+iS^*}{C^*}$ and $\alpha = e^{-2\beta} = \tanh\beta^*$. At the critical point $\beta = \beta_c$ ($C^* = C_c^* = \sqrt{2}$ and $S^* = S_c^* = 1$, so $\tau = \tau_c = \lambda^3$), these equations reproduce equations (131). Moreover, it has been shown in [BeDC10] that the massive s-holomorphicity implies the massive Laplace equation,

$$\frac{1}{4}\sum_{Z=X\pm 1, X\pm\mathrm{i}} (F(Z) - F(X)) = \mu F(X),$$

for X is the bulk point of the lattice and $\mu = \frac{S+S^{-1}}{2} - 1$. For further discussion see section (2.2) in [HKZ12].

Both equations (131) and (132) imply that we have four real-linear equations and two complex unknowns (four linear unknowns) per plaquette. Therefore, we have roughly equal number of equations and unknowns in the lattice. As we mentioned, s-holomorphicity relations are stronger notion of discrete analyticity eq. (129). In fact, s-holomorphicity equations are enough to determine the values of the function at every bulk points of the lattice in terms of the boundary points. As we will see in the "propagation of s-holomorphic functions" part in chapter (4), the set of s-holomorphicity equations can be used to write for example the value of the function in an specific column and row in the bulk of the lattice in terms of the values of the functions in previous row and one column before and after. Moreover, in the next part, we fix the phases of the functions on the four boundaries of the lattice which is called Riemann-Hilbert boundary value problem. By using the s-holomorphicity relations and values of the function at boundaries, the function is uniquely determined on the bulk points of the lattice as well as boundary points.

THE RIEMANN-HILBERT BOUNDARY VALUE PROBLEM FOR S-HOLOMORPHIC FUNCTIONS

In order to study the Ising model on the square lattice with specific boundary conditions we will use s-holomorphic functions as concrete mathematical tools. In addition to the above equations, the s-holomorphic functions of the Ising model satisfy boundary conditions that specify the argument of the functions on the boundaries. Therefore, they are solutions of the *Riemann-Hilbert boundary condition* as defined in the following.

A function $F\colon D_\diamond \to \mathbb{C}$ satisfies the Riemann-Hilbert boundary conditions, if at all boundary midpoints of edges the value of the function is determined by the type of the boundary edge in fig. (1) as follows:

-For type N the value is purely real, $F(N) \in \mathbb{R}$;

-For type S the value is purely imaginary, $F(S) \in i\mathbb{R}$;

-For type W the value is a real multiple of $\lambda = e^{\mathrm{i}\pi/4}$, $F(W) \in \lambda\mathbb{R}$;

-For type E the value is a real multiple of λ^{-1}, $F(E) \in \lambda^{-1}\mathbb{R}$.

As it is stated and proved in section (2.6) in [Hon10a], any discrete Riemann-Hilbert boundary value problem has at most one solution and if we assume the existence of the solution, then the Riemann-Hilbert boundary value problem is a well posed problem with a unique solution.

3.2.2 PARAFERMIONIC OBSERVABLES

The next step in our study is to explicitly determine the s-holomorphic functions of the Ising model which satisfy the s-holomorphicity conditions as well as Riemann boundary conditions except than some exceptional points. These solutions are called s-holomorphic parafermionic or winding observables. In fact, there are different winding observables for critical lattice Ising model which satisfy s-holomorphicity conditions as well as Riemann boundary conditions except than some exceptional points. In this section, we try to give a short introduction to these parafermionic observables and their physical meanings.

As we mentioned, the parafermionic observables have played an important role in exact studies about the convergence to conformal field theory in scaling limit of the $2d$ lattice models at critical temperature. Specially, in two directions; i) the rigorous proofs of the convergence of lattice models interfaces to the SLE conformal curves and ii) the proof of convergence of various correlation functions of the critical lattice model in the continuum limit to CFT correlation functions.

For Smirnov, the first important use of s-holomorphicity was to show conformal invariance of the scaling limit of two particular observables that pertain to the proofs that macroscopic spin-cluster boundaries and Fortuin-Kasteleyn-cluster boundaries tend in the scaling limit to two different SLE type random curves.

More recently, s-holomorphic observables have been used to obtain rigorous results about the conformal invariance of the Ising model somewhat more directly. In [HoSm10b], the scaling limit of the energy density of the critical Ising model is derived from an s-holomorphic observable, and in [Hon10a] the method is generalized to the derivation of any energy correlations and also boundary spin correlations. Finally, the physically very important spin correlations in the bulk were shown to have conformally invariant scaling limits in the works [ChIz11] and [CHI12], using s-holomorphic observables that are multi-valued and branching around singularities.

SMIRNOV OBSERVABLE

The first kind of the Ising parafermionic observable on the rectangular lattice is introduced by S. Smirnov, [Smi06], [Smi10a] and [ChSm09]. It is a graphical expansion on the rectangular lattice and it has one boundary condition changing point a on the bottom of the lattice and one bulk point z. It is defined by the following expansion

$$F_a(z) = \frac{1}{\mathfrak{Z}} \sum_{\sigma \in \mathcal{C}_{a,z}} \alpha_c^{L(\sigma)} e^{-\frac{i}{2} w(\sigma : a \rightsquigarrow z)}, \tag{133}$$

where $\mathfrak{Z}$ is a partition function, the sum is over collections of dual edges in all graphical expansions $\sigma \in \mathcal{C}_{a,z}$, consisting of loops and a path from a to z, $\alpha_c = e^{-2\beta_c}$ and w is the winding number of directed path starting upward at a and ending at z either from above or below. The point a is a midpoint of a horizontal edge at the bottom boundary of the lattice and the point z is a midpoint of a horizontal or vertical edge in the bulk of the lattice. A partition function of the critical model with plus boundary conditions is defined by

$$\mathfrak{Z} = \sum_{\sigma \in \mathcal{C}^+} \alpha_c^{L(\sigma)}, \tag{134}$$

where the sum is over collections of dual edges in all configurations with plus boundary conditions, $\sigma \in \mathcal{C}^+$ and $L(\sigma)$ is number of edges in each configuration $\sigma \in \mathcal{C}^+$.

Smirnov observable is one of the appropriate tools to study the scaling limit of the critical $2d$ Ising model. All the exact results about the convergence of the scaling limit of the critical Ising model to conformal field theory, including the conformal invariance of the scaling limit of the interfaces are obtained by using these observables. Specially the convergence of interfaces to SLE_3 is obtained by implementing this method.

ENERGY OBSERVABLE AND PFAFFIAN FORMULAS

The energy density observable is defined by Hongler and Smirnov [Hon10a] and [HoSm10b]. It has two bulk points z, z' at midpoint of the edges in the configurations and it's defined by

$$F_{z'}(z) = \frac{1}{\mathfrak{Z}} \sum_{\sigma \in \mathcal{C}_{z',z}} \alpha_c^{L(\sigma)} e^{-\frac{i}{2} w(\sigma : z' \rightsquigarrow z)}, \tag{135}$$

where the sum is over collections of dual edges in all configurations $\sigma \in \mathcal{C}_{z',z}$, consisting of loops and a directed path between z' and z starts at z' either from above or below and ending at z either from above or below. The relation between this definition and the energy density definition of the Ising model has been explored in [HoSm10b].

Notice that when the z' is on the bottom boundary of the lattice, the energy density observables reduces to the Smirnov observables. The energy density observable $F_{z'}(z)$ at $z' = a$, the bottom boundary of the lattice, is the solution of the Riemann boundary value problem and it is the appropriate observable to study the convergence of the energy correlation functions of the critical Ising model in the scaling limit.

The definition of the energy density observable can be generalized to a multi-point parafermionic observable,

$$F(z_1^{\eta_1},...,z_{2n}^{\eta_{2n}}) = \frac{1}{\mathfrak{Z}} \sum_{\sigma\in\mathcal{C}_{z_1,...,z_{2n}}} \alpha_c^{L(\sigma)} e^{-\frac{\mathrm{i}}{2}\sum_{\{z_i,z_j\}} w(\sigma:z_i\rightsquigarrow z_j)} Sgn(p), \tag{136}$$

where the sum is over collections of dual edges in all configurations $\sigma \in \mathcal{C}_{z_1,...,z_{2n}}$, consisting of loops and n oriented paths linking the bulk points $z_1,...,z_{2n}$ and $\eta_i = \pm 1$ depends on the possibilities to start at z_i from above or below. The notation $\{z_i, z_j\}$ denotes the path between z_i and z_j. The sign factor $Sgn(p) = (-1)^{n_c}$ is the sign of pairing of the paths and n_c is the number of crossing in the pairing of the paths. There is a relation between the multi-point functions and two-point functions which is called *Pfaffian formula*. The Pfaffian formula for the generalized energy density observable is obtained in [Hon10a] and it can be stated as follow

$$F(z_1^{\eta_1},...,z_{2n}^{\eta_{2n}}) = Pf\left[\left(F(z_i^{\eta_i} z_j^{\eta_j})\right)_{i,j=1}^{2n}\right], \tag{137}$$

where the definition of the Pfaffian of an anti-symmetric matrix $A \in \mathbb{C}^{n\times n}$ is

$$Pf(A) = \begin{cases} \frac{1}{2^k k!}\sum_P sgn(P)\prod_{i=1}^k A_{P(2i-1),P(2i)} & \text{for } n = 2k \\ 0 & \text{for } n = 2k-1 \end{cases}. \tag{138}$$

SPINOR OBSERVABLE AND BRANCHING AT SPIN INSERTION

The spinor observables are defined by Chelkak, Hongler and Izyurov [ChIz11], [CHI12]. They are multi-valued parafermionic observables on the Riemann surface, double cover of the domain, which have spin operators,

$$F_a^{spinor}(z,z') = \frac{1}{\mathfrak{Z}_s} \sum_{\sigma\in\mathcal{C}_{spinor}} \alpha_c^{L(\sigma)} e^{-\frac{\mathrm{i}}{2}w(\sigma:a\rightsquigarrow z)}(-1)^{l_{z'}(\sigma)}(-1)^{I(\sigma)}, \tag{139}$$

where $\mathfrak{Z}_s = \sum_{\sigma\in\mathcal{C}_s^+} \alpha_c^{L(\sigma)}(-1)^{l_{z'}(\sigma)}$, in which the sum is over all graphical expansions with plus boundary conditions and one spin insertion, the sum in eq. (139) is over collections of dual edges in all configurations $\sigma \in \mathcal{C}_{spinor}$, consisting of loops and a path from a to z, $l_{z'}(\sigma)$ is number of loops surrounding spin operator at point z' and $I(\sigma)$ indicates that the endpoint of the path, z, is on principal sheet or opposite sheet,

$$I(\sigma) = \begin{cases} 0 & \textit{endpoint at } z \\ 1 & \textit{endpoint at } \tilde{z} \end{cases}, \tag{140}$$

where $\tilde{z}$ is the corresponding point of z on the opposite sheet.

The spinor observable on the lower sheet of the double cover is minus the spinor observable on the upper sheet. That is a property that we expect from the fermionic observables. The spinor observable can be simply generalized to include more than one spin operator as

$$F_a^{spinor}(z; z_1',...,z_S') = \frac{1}{\mathfrak{Z}_S} \sum_{\sigma\in\mathcal{C}_{Spinor}} \alpha_c^{L(\sigma)} e^{-\frac{\mathrm{i}}{2}w(\sigma:a\rightsquigarrow z)} \prod_{s=1}^{S}(-1)^{L_{z_s'}(\sigma)}, \tag{141}$$

where $\mathfrak{Z}_S = \sum_{\sigma\in\mathcal{C}_S^+} \alpha_c^{L(\sigma)}\prod_{s=1}^S(-1)^{l_{z_s'}(\sigma)}$, in which the sum is over all configurations with plus boundary conditions and many spin insertions, the sum in eq. (141) is over collections of dual edges in all graphical expansions $\sigma \in \mathcal{C}_{Spinor}$, consisting of loops and a path from a to z, z_s' is a point on a vertices of the lattice, $L_{z_s'}(\sigma)$ is the number of the vertical edges on the right hand side of the spin operator $\hat{\sigma}_{z_s'}$.

Because spinor observables with appropriate partition function are solutions to the discrete Riemann boundary value problem they provide the mathematical tools to study the scaling limit of the spin operators correlation functions in the critical $2d$ Ising model.

4 Ising fermions, discrete holomorphicity and SLE: summary of results

In the first part of this chapter we investigate the relations between the old powerful picture of fermionic techniques and the new rigorous discrete holomorphicity methods in two directions.

First, the connections between properties and behaviors of fermion operators and s-holomorphic functions are investigated. We consider the fermion operators $\psi(z)$ and $\bar{\psi}(z)$ acting on the fermionic Fock space, indexed by z a midpoint of a horizontal edge of the square lattice domain. These are first order expressions in the generators of the Clifford algebra. We find that, inside and outside of the critical temperature, the action of induced rotation of transfer matrix on fermions of Ising model leads to time propagation of fermions on the lattice and this propagation relation is roughly the same as the relations between s-holomorphic functions on the lattice, derived from s-holomorphicity conditions. Second, we show that the fermionic correlation functions are closely related to s-holomorphic observables.

Beside the understanding of the deep relations between the two topics, there are new advantages such as proof of Pfaffian formula for s-holomorphic multi-point correlations by using Wick's formula, and a generalization of fermion operator definition.

Then, we consider two-dimensional fermionic CFT or the theory of free fermions. We will find that this is an appropriate theory to describe the scaling limit of the $2d$ Ising model at critical temperature. The exact rigorous proofs of the conformal invariance of scaling limit of critical lattice models was missing until recently. Smirnov's results provide the proof of the conformal invariance and convergence of boundaries to some mathematical objects, SLE curves. In this direction, we study the fermionic conformal field theory, which describe the scaling limit of critical Ising model, and its relations to the SLE_3 curves as the interfaces appearing in the scaling limit of the Ising model at critical temperature.

4.1 Free fermions/S-holomorphicity correspondence

In this section we present our main results of studies about Ising free fermions and s-holomorphic functions, [HKZ12]. We will observe that the fermion operators and s-holomorphic functions have similar behavior and properties. For instance, they both have the same row-to-row propagation and the same spectra etc. Moreover, there are close relations between fermion correlations and parafermionic observables. All of these results generalize to the case of massive s-holomorphic functions and Ising model transfer matrix outside of the critical temperature.

Furthermore, we have obtained some results such as i) proof of Pfaffian formula for (massive) s-holomorphic functions by using the easy and powerful method of Wick's formula for multi-point fermions correlations, ii) extension of the definition of fermion operator to the midpoint of vertical edges by using the definition of s-holomorphicity and iii) an algebraic construction of rigorous scaling limit of transfer matrix.

4.1.1 Local relations, propagations and spectra

An extension of the fermion operator to half-integer rows

In previous sections, we have defined fermion operators $\psi(z)$ and $\bar{\psi}(z)$, acting on the fermionic Fock space, at z, mid-points of horizontal edges. However, we can extend this definition uniquely in such a way that fermion operators take value on mid-points of vertical edges as well. As we have seen, s-holomorphic functions are defined on mid-points of all edges including vertical and horizontal. The extensions satisfy linear relations among neighboring midpoints of edges in the bulk with coefficients that coincide with the defining relations of s-holomorphicity. Moreover, the extensions satisfy linear relations at midpoints of boundary edges with coefficients that coincide with the boundary relations defining the discrete Riemann-Hilbert boundary value problem. Notice that, in general using massive s-holomorphicity leads to a temperature dependent definition. It can be seen from local relations of fermions.

Local relations of fermions

For the first time, the linear equations between lattice fermions were found for the Ising model in [DoPo88]. In our study, we found that the pair of Ising fermion operators $\psi(z)$ and $\bar{\psi}(\bar{z})$ form a complexified operator-valued s-holomorphic function solving the Riemann-Hilbert boundary value problem. In the following, two precise formulations of this result are given in terms of local relations and the propagation of values of functions from one row to the next.

From the s-holomorphicity conditions (131) and (132), in the massive as well as massless case, we can essentially find the local relations between fermions as follow;

$$\begin{aligned}\psi(N)+\lambda\bar{\psi}(N)&=\psi(E)+\lambda\bar{\psi}(E)\\ \psi(N)+\lambda^{-1}\bar{\psi}(N)&=\psi(W)+\lambda^{-1}\bar{\psi}(W)\\ \psi(S)+\lambda^{3}\bar{\psi}(S)&=\psi(E)+\lambda^{3}\bar{\psi}(E)\\ \psi(S)+\lambda^{-3}\bar{\psi}(S)&=\psi(W)+\lambda^{-3}\bar{\psi}(W),\end{aligned}\tag{142}$$

and

$$\begin{aligned}\tau\psi(N)-\bar{\psi}(N)&=\lambda^{3}\psi(E)+\tau\lambda\bar{\psi}(E)\\ \psi(N)-\tau\bar{\psi}(N)&=\tau\lambda^{-3}\psi(W)+\lambda^{-1}\bar{\psi}(W)\\ \psi(S)+\tau\bar{\psi}(S)&=\tau\lambda^{-3}\psi(E)+\lambda^{3}\bar{\psi}(E)\\ \tau\psi(S)+\bar{\psi}(S)&=\lambda^{3}\psi(W)+\tau\lambda^{-3}\bar{\psi}(W),\end{aligned}\tag{143}$$

where λ and τ are defined in section (3.2.1). Furthermore, there are local relations for the fermions on the left and right boundaries

$$\psi(a+\mathrm{i}m)+\mathrm{i}\bar{\psi}(a+\mathrm{i}m)=0,\quad \psi(b+\mathrm{i}m)-\mathrm{i}\bar{\psi}(b+\mathrm{i}m)=0,\tag{144}$$

where $a=-M+\frac{1}{2}, b=M-\frac{1}{2}$ and for any m. A detailed rigorous proof of the above results are given in theorem (19) of [HKZ12].

Furthermore, it can be guessed that in principle, there is an appropriate discretization scheme for the Dirac equation which gives the above results.

Row-to-row propagations of fermions and s-holomorphic functions

In the following, first we obtain the propagation of fermions by using the definition of fermion, in terms of the Clifford algebra generators p and q, and the induced rotations of the Clifford algebra generators. Then, we obtain the propagation of s-holomorphic functions by using the s-holomorphicity definition. Finally, we compare the obtained results for propagation of the fermions and s-holomorphic functions.

Propagations of fermions

In this section we state important results, namely the row-to-row propagation of the lattice fermions, theorem (10) in [HKZ12]. The starting point of the calculation is the definition of time-dependent fermion operators as follow:

$$\psi(k+\mathrm{i}m)=V^{-m}\psi_k V^{m},\qquad \bar{\psi}(k+\mathrm{i}m)=V^{-m}\bar{\psi}_k V^{m}.\tag{145}$$

Then the following relations define the row-to-row propagation,

$$\begin{aligned} \psi(k+\mathrm{i}(m+1)) &= V^{-m-1}\psi_k V^{m+1} = T(V)^{-1}\psi(k+\mathrm{i}m) \\ &= V^{-1}\psi(k+\mathrm{i}m)V = V^{-m}(V^{-1}\psi_k V)V^m. \end{aligned} \tag{146}$$

The induced rotations of the p_k and q_k are obtained in lemma in section (3.2) in [HKZ12], from the eqs. (108) and (109) by using the Taylor series expansion

$$\exp(\alpha X) v \exp(-\alpha X) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}[X, ..., [X, v], ...]. \tag{147}$$

For simplicity in the incoming results, we will use the following short hands for the values of hyperbolic functions

$$\begin{aligned} c &= \cosh(\beta), & s &= \sinh(\beta), \\ C &= \cosh(2\beta) = c^2 + s^2, & S &= \sinh(2\beta) = 2cs, \end{aligned}$$

and

$$\begin{aligned} c^* &= \cosh(\beta^*), & s^* &= \sinh(\beta^*), \\ C^* &= \cosh(2\beta^*), & S^* &= \sinh(2\beta^*). \end{aligned}$$

We collect the useful formulas for induced rotation of p and q in the bulk and the boundary of the lattice, see section (3.2) in [HKZ12].

The action of V_i on boundaries:

$$V_i p_{-M-\frac{1}{2}} V_i^{-1} = p_{-M-\frac{1}{2}},$$

$$V_i q_{M+\frac{1}{2}} V_i^{-1} = q_{M+\frac{1}{2}},$$

for $i = 1, 2$.

The action of V_1 in bulk:

$$V_1 q_l V_1^{-1} = Cq_l - \mathrm{i}Sp_l,$$

$$V_1 p_l V_1^{-1} = \mathrm{i}Sq_l + Cp_l,$$

for $l = -M + \frac{1}{2}, ..., M - \frac{1}{2}$.

The action of V_2 in bulk:

$$V_2 q_{-M+\frac{1}{2}} V_2^{-1} = q_{-M+\frac{1}{2}},$$

$$V_2 p_{M-\frac{1}{2}} V_2^{-1} = p_{M-\frac{1}{2}}.$$

And finally for $l = -M + \frac{1}{2}, ..., M - \frac{3}{2}$:

$$V_2 q_{l+1} V_2^{-1} = \mathrm{i}S^* p_l + C^* q_{l+1},$$

$$V_2 p_l V_2^{-1} = C^* p_l - \mathrm{i}S^* q_{l+1}.$$

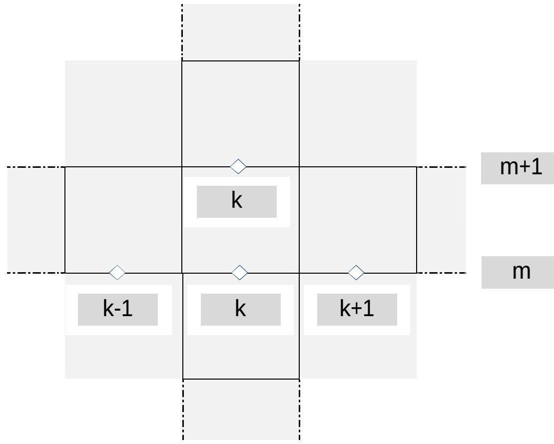

Figure 2: Bulk propagation of fermions

BULK PROPAGATIONS

The induced rotation of the p_k and q_k in the bulk of the lattice can be easily calculated as follow:

$$
\begin{aligned}
V^{-1}p_kV &= V_1^{-1/2}V_2^{-1}V_1^{-1/2}p_kV_1^{1/2}V_2V_1^{1/2}\\
&= C^*Cp_k - \mathrm{i}C^*Sq_k + \mathrm{i}S^*c^2q_{k+1} - \frac{1}{2}p_{k+1} - \frac{1}{2}p_{k-1} + \mathrm{i}S^*s^2q_{k-1},
\end{aligned}
\tag{148}
$$

and

$$
\begin{aligned}
V^{-1}q_kV &= V_1^{-1/2}V_2^{-1}V_1^{-1/2}q_kV_1^{1/2}V_2V_1^{1/2}\\
&= C^*Cq_k + \mathrm{i}C^*Sp_k - \frac{1}{2}q_{k+1} - \mathrm{i}S^*s^2p_{k+1} - \mathrm{i}S^*c^2p_{k-1} - \frac{1}{2}q_{k-1}.
\end{aligned}
\tag{149}
$$

Then by using the definition $\psi_k = A_\psi(q_k + p_k)$ and the induced rotations of the p_k and q_k in the bulk of the lattice in eqs. (148) and (149) we obtain the induced rotation for the fermion operator in the bulk of the lattice at arbitrary temperature, see fig (2),

$$
\begin{aligned}
V^{-1}\psi_kV &= C^*C\psi_k + \left(-\frac{1}{2} - \frac{\mathrm{i}}{2}S^*\right)\psi_{k-1} + \left(-\frac{1}{2} + \frac{\mathrm{i}}{2}S^*\right)\psi_{k+1}\\
&\quad + \frac{A_\psi}{A_{\bar\psi}}\left(\mathrm{i}C^*S\bar\psi_k - \frac{\mathrm{i}}{2}S^*C\bar\psi_{k-1} - \frac{\mathrm{i}}{2}S^*C\bar\psi_{k+1}\right).
\end{aligned}
\tag{150}
$$

At the critical temperature or equivalently for massless fermion we obtain

$$
\begin{aligned}
V^{-1}\psi_kV &= 2\psi_k + \frac{-\mathrm{i}-1}{2}\psi_{k-1} + \frac{\mathrm{i}-1}{2}\psi_{k+1}\\
&\quad + \frac{A_\psi}{A_{\bar\psi}}\left(\sqrt{2}\mathrm{i}\bar\psi_k - \frac{\mathrm{i}}{\sqrt{2}}\bar\psi_{k-1} - \frac{\mathrm{i}}{\sqrt{2}}\bar\psi_{k+1}\right).
\end{aligned}
\tag{151}
$$

Similarly, we can obtain the propagation of fermions on the boundaries of lattice.

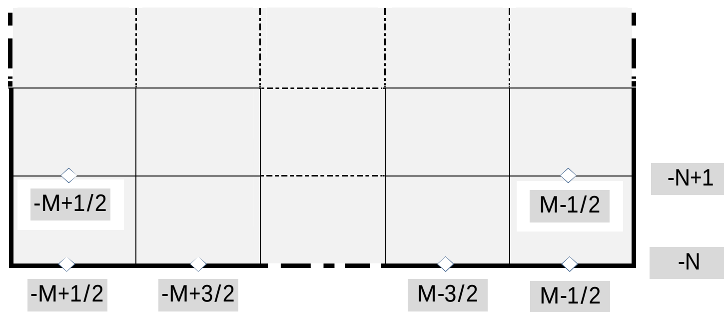

Figure 3: Boundary propagation of fermions

BOUNDARY PROPAGATIONS

On the left and right boundaries we have slightly different induced rotations. For example, on the left boundary we have

$$\begin{aligned} V^{-1}p_{-M+\frac{1}{2}}V &= V_1^{-1/2}V_2^{-1}V_1^{-1/2}p_{-M+\frac{1}{2}}V_1^{1/2}V_2V_1^{1/2} \\ &= (cC^*c+s^2)p_{-M+\frac{1}{2}} - \mathrm{i}c(C^*+1)sq_{-M+\frac{1}{2}} \\ &\quad -cS^*sp_{-M+\frac{3}{2}} + \mathrm{i}cS^*cq_{-M+\frac{3}{2}}, \end{aligned}$$

and

$$\begin{aligned} V^{-1}q_{-M+\frac{1}{2}}V &= V_1^{-1/2}V_2^{-1}V_1^{-1/2}q_{-M+\frac{1}{2}}V_1^{1/2}V_2V_1^{1/2} \\ &= \mathrm{i}(cs+sC^*c)p_{-M+\frac{1}{2}} + (c^2+sC^*s)q_{-M+\frac{1}{2}} \\ &\quad -\mathrm{i}sS^*sp_{-M+\frac{3}{2}} + sS^*cq_{-M+\frac{3}{2}}. \end{aligned}$$

In a similar way to the case of bulk, for the induced rotation of fermion operator on the left boundary in fig. (3) we obtain

$$\begin{aligned} V^{-1}\psi_{-M+\frac{1}{2}}V &= \frac{(1+C^*)C}{2}\psi_{-M+\frac{1}{2}} + \frac{(S^*+\mathrm{i})}{2}\mathrm{i}\psi_{-M+\frac{3}{2}} \\ &\quad +\frac{A_\psi}{A_{\bar\psi}}\left(\frac{(C^*-1)+\mathrm{i}(S+C)}{2}\bar\psi_{-M+\frac{1}{2}} - \frac{C^*}{2}\mathrm{i}\bar\psi_{-M+\frac{3}{2}}\right). \end{aligned} \tag{152}$$

And at the critical limit, for the massless fermion, this becomes

$$\begin{aligned} V^{-1}\psi_{-M+\frac{1}{2}}V &= (1+\frac{1}{\sqrt{2}})\psi_{-M+\frac{1}{2}} + \frac{\mathrm{i}-1}{2}\psi_{-M+\frac{3}{2}} \\ &\quad +\frac{A_\psi}{A_{\bar\psi}}\left(\frac{1}{2}\left((\sqrt{2}-1)+\mathrm{i}(\sqrt{2}+1)\right)\bar\psi_{-M+\frac{1}{2}} - \frac{\mathrm{i}}{\sqrt{2}}\bar\psi_{-M+\frac{3}{2}}\right). \end{aligned} \tag{153}$$

Similarly, for fermions on the right boundary there are similar relations.

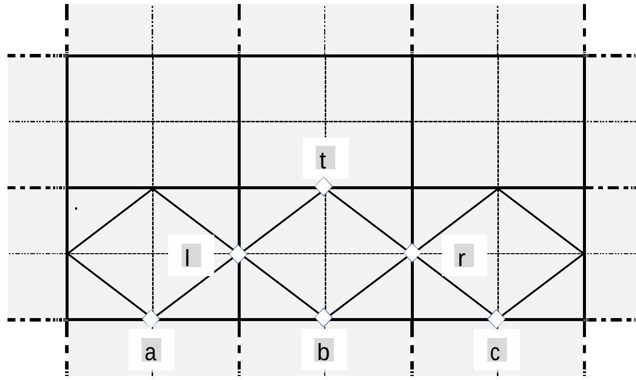

Figure 4: Bulk propagation of s-holomorphic functions

Propagation of s-holomorphic functions

Having introduced the s-holomorphic functions, we discuss the concept of propagation of these functions which is defined based on the s-holomorphicity local relations. Consider the restrictions of an s-holomorphic function to one horizontal line, then by propagation we mean how the values of the function change from one line to the next. In the case of a rectangle $\Lambda_{M,N}$, we call it *"row-to-row propagation"*. The space of possible values in one row of $\Lambda_{M,N}$ is denoted by

$$\mathcal{S}_M(row) = \mathbb{C}^{[[-M,M]]^*},$$

where $[[-M, M]]^*$ and $[[-M, M]]$ denote the half-integer and integer interval from $-M$ to M, respectively. For example in the row $m \in [[-N, N]]$, the $f_m \in \mathcal{S}_M(row)$ is determined by

$$f_m(k) = F(k + \mathrm{i}m), \qquad k \in [[-M, M]]^*.$$

We will define an $\mathbb{R}$-linear operation, $P : \mathcal{S}_M(row) \to \mathcal{S}_M(row)$ such that the s-holomorphic functions with Riemann-Hilbert boundary values, f_{m+1}, are determined by

$$f_{m+1} = P\, f_m.$$

The spectrum and eigenvectors of the operator P, are intimately related to the asymptotics of s-holomorphic functions since

$$F(k + \mathrm{i}m) = \left(P^m\, f_0\right)(k).$$

For further details see section (2.4) in [HKZ12].

Bulk propagation

Let us consider the propagation of s-holomorphic functions in the bulk of the lattice as demonstrated in fig. (4). To write down explicit formulas for $F(t)$ in terms of $F(l)$ and $F(r)$, and then for $F(l)$ and $F(r)$ in terms of $F(a)$, $F(b)$, $F(c)$, one solves the linear systems given by the defining eqs. (131) and (132). The final expression for the row-to-row propagation of the massive s-holomorphic function can be obtained as

$$\begin{aligned} F(t) = {} & \frac{-1 - \mathrm{i}S^*}{2} F(a) + C^* C\, F(b) + \frac{-1 + \mathrm{i}S^*}{2} F(c) \\ & + \frac{C^*}{2}\overline{F(a)} - C\overline{F(b)} + \frac{C^*}{2}\overline{F(c)}. \end{aligned}$$

At the critical point $\beta = \beta_c$ this becomes

$$F(t) = \frac{\lambda^{-3}}{\sqrt{2}}\, F(a) + 2\, F(b) + \frac{\lambda^3}{\sqrt{2}}\, F(c) + \frac{1}{\sqrt{2}}\, \overline{F(a)} - \sqrt{2}\, \overline{F(b)} + \frac{1}{\sqrt{2}}\, \overline{F(c)}.$$

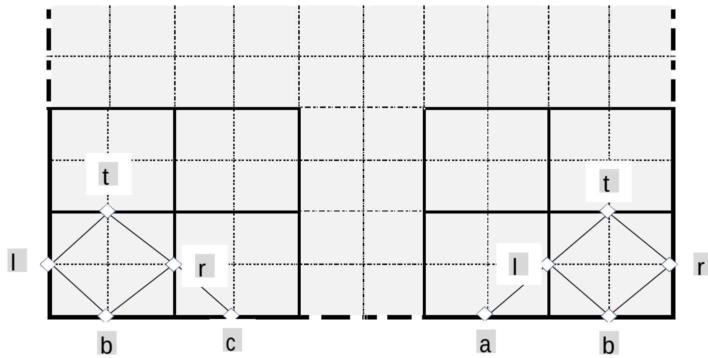

Figure 5: Boundary propagation of s-holomorphic functions

BOUNDARY PROPAGATION

On the boundary of the rectangle, the row-to-row propagation of s-holomorphic functions is not determined only by s-holomorphicity relations, but we also need to consider the boundary conditions as it is defined in Riemann-Hilbert boundary value problem.

For example let us consider the left boundary demonstrated in fig. (5). The row-to-row propagation for the left boundary is the expression for $F(t)$ in terms of $F(b)$ and $F(c)$ as follow:

$$F(t) = \frac{(1+C^*)C}{2} F(b) + \frac{-1+\mathrm{i}S^*}{2} F(c) + \frac{-(1+C^*)S + \mathrm{i}(C^*-1)}{2} \overline{F(b)} + \frac{C^*}{2} \overline{F(c)}.$$

At critical point $\beta = \beta_c$ this becomes

$$F(t) = \left(1 + \frac{1}{\sqrt{2}}\right) F(b) + \frac{\lambda^3}{\sqrt{2}} F(c) + \left(\lambda^3 + \frac{\lambda^{-3}}{\sqrt{2}}\right) \overline{F(b)} + \frac{1}{\sqrt{2}} \overline{F(c)}.$$

The explicit form of the operator P in massive and massless cases are given in lemma (6) in [HKZ12].

We observe that the row-to-row propagations of s-holomorphic functions are closely related to the propagations of fermions by the action of induced rotation of the transfer matrix.

In the following we present the main results of this part, theorem (10) and theorem (18) in [HKZ12]. The theorems are proved rigorously in the paper and can be summarized as follow. In the Ising model on rectangular lattice with plus or minus boundary conditions and with the choice $\frac{A_\psi}{A_{\bar{\psi}}} = \mathrm{i}$, the row-to-row propagations of the free massive fermions, obtained from conjugations by the transfer matrix in this section, become the complexification of row-to-row propagation of the massive s-holomorphic functions, $P^{\mathbb{C}}$, that satisfy massive s-holomorphicity conditions as well as Riemann boundary conditions. Especially, there exist a linear isomorphism $\rho : (\mathbb{C}^2)^{[[-M,M]]^*} \to W_M$ such that

$$T(V_M) = \rho \circ P^{\mathbb{C}} \circ \rho^{-1}. \tag{154}$$

In another words, a complexification of the space $\mathcal{S}_M(row)$ of values of functions in a row can be identified with the space of first order elements of the Clifford algebra. At $\beta = \beta_c$, the complexification of the row-to-row propagation of s-holomorphic solutions to the Riemann-Hilbert boundary value problem becomes row-to-row propagations of free massless fermions.

SPECTRUM OF PROPAGATION

It can be shown analytically that the operator P implementing the row-to-row propagation of s-holomorphic solutions of the Riemann-Hilbert boundary value problem is diagonalizable with positive eigenvalues, Λ, Λ^{-1} and $\Lambda \neq 1$, proposition (7) in [HKZ12].

Let us briefly summarize the Bethe ansatz approach in the analysis of the eigenvectors and eigenvalues of the induced rotation of the transfer matrix $T(V_M)|_{W_M} = T_M = T(V_1^{\frac{1}{2}})T(V_2)T(V_1^{\frac{1}{2}})$, [Pal07]. First, let us define the spectral parameter $z = |z|e^{\mathrm{i}\omega}$ and $\varphi_i(\omega) := arg\frac{e^{\mathrm{i}\omega}-\alpha_i}{1-\alpha_i e^{\mathrm{i}\omega}}$, where $\alpha_1 = e^{2(\beta-\beta^*)} = (C^*-S^*)(C+S)$ and $\alpha_2 = e^{2(\beta+\beta^*)} = (C^*+S^*)(C+S)$.

Theorem (Theorem (1.2.2) in [Pal07]): Suppose that $\alpha_1 > 1$, $T < T_c$. Then, the spectrum of T_M consists of $4M$ distinct values $e^{\pm\gamma_n}$ for $n = 1, 2, ..., 2M$, where $\gamma_n = \gamma(e^{\mathrm{i}\omega_n})$ and $\gamma(z) \geq 0$ is given by

$$\cosh\gamma(z) = C^*C - S^*S\frac{z+z^{-1}}{2}, \tag{155}$$

and ω_n is the positive root of

$$2M\omega - n\pi = \frac{\varphi_1(\omega)+\varphi_2(\omega)}{2}, \tag{156}$$

lying between 0 an π. This is obtained from the eigenvalue equation

$$z^{4M} = \frac{z-\alpha_1}{1-\alpha_1 z}\frac{z-\alpha_2}{1-\alpha_2 z}. \tag{157}$$

Moreover, the eigenvector, S^1-valued function $v(z)$, satisfies

$$\begin{bmatrix}\cosh\gamma & v\sinh\gamma\\ \bar{v}\sinh\gamma & \cosh\gamma\end{bmatrix}\begin{bmatrix}\pm v\\ 1\end{bmatrix} = e^{\pm\gamma}\begin{bmatrix}\pm v\\ 1\end{bmatrix}, \tag{158}$$

$$v(z)\sinh\gamma(z) = \mathrm{i}\left(S^*C - C^*S\frac{z+z^{-1}}{2}\right). \tag{159}$$

Furthermore, it is shown that

$$\cosh\gamma(e^{\mathrm{i}\omega}) \geq \cosh(2\beta-2\beta^*) > 1. \tag{160}$$

We have explicitly checked that the eigenvalues and eigenvectors of operator P both coincide with the results of the analysis of the Ising model spectrum in [Pal07].

4.1.2 Parafermionic observables and fermions correlation functions

Particular correlation functions of the fermion operators reproduce the s-holomorphic observables that are used in the recent rigorous approaches to the conformal invariance of the Ising model. Precisely, case by case it can be proved that the various s-holomorphic observables are consistently related to the low-temperature expansion of the fermion, fermion-antifermion and spin-fermion correlations. Furthermore, for any n-point fermion correlation function there is a corresponding parafermionic observable, chapter (4) in [HKZ12]. In the following first we derive the graphical expansions of the fermionic correlation functions and then we compare the obtained results with the parafermionic observables. This equivalency between the fermionic correlation functions and parafermionic observables provides a concrete and explicit connection between the standard physical results in Ising model and the new rigorous method of s-holomorphic observables in mathematical studies of Ising model.

Low temperature expansions for fermions

In the low-temperature expansion, each configuration includes collections of spins up and down in different regions with some edges surrounding those regions such as loops. Moreover, in correlation functions, there are also operators such as spin or fermion inserted on the lattice vertices or midpoints of edges. The paths start at fermion insertion move along interfaces and end at another fermion insertion points. The fermion

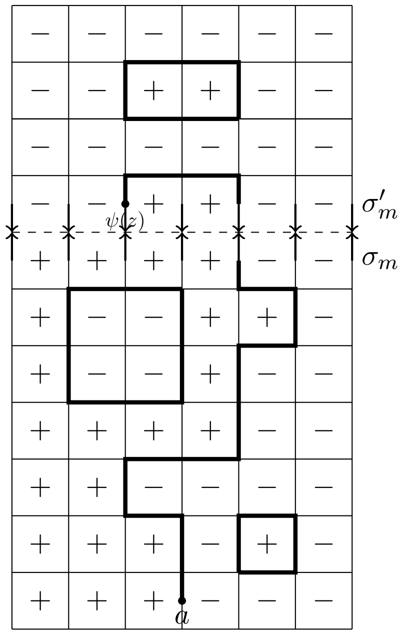

Figure 6: One-point fermion correlation function I

operator is located either on the boundaries or in the bulk. Depending on the correlation functions, the configurations include different operators and they have different graphical representations.

Having introduced the low-temperature expansion in Ising model, in the following, the low temperature expansions for single and multi-point correlation functions such as fermion, fermion-antifermion, spin-fermion and higher point correlation functions at critical temperature are obtained.

EXPANSIONS OF ONE-POINT AND TWO-POINT FERMION CORRELATIONS

In the first part of this section, we define the low-temperature expansion of a single fermion correlation function with an specific boundary conditions, with one boundary change which means that at one point on the lower boundary of the lattice, the sign of the spin changes. Let us define the states, $|++>= V^N|e_{(+)}^{-N}>$, $<++|=<e_{(+)}^{N}|V^N$ and $e_{(+)}$ is the vector corresponding to a $(+)$ row configuration at an arbitrary row in which all the spins are plus. All the other states such as $|+->$ are defined similarly. We describe the correlation function $<--|\psi(z)|+->/Z$, where z is on the discrete complex plane, $z=k+\mathrm{i}m$, and the normalization factor is $Z=<++|++>$.

Let start by one-point fermion correlation function which is written in a series expansion as follow

$$
\begin{aligned}
<--|\psi(k+\mathrm{i}m)|+->= \sum_{\sigma_i\in\mathcal{C}_\Lambda(\text{row})} \quad & V_{\sigma_{-N+1},+-}V_{\sigma_{-N+2},\sigma_{-N+1}}\cdots \\
& V_{\sigma_m,\sigma_{m-1}}\psi k_{\sigma'_m,\sigma_m}V_{\sigma_{m+1},\sigma'_m}\cdots \\
& V_{\sigma_{N-1},\sigma_{N-2}}V_{--,\sigma_{N-1}}, \qquad (161)
\end{aligned}
$$

where the sum is over all possible row configurations $\sigma_i\in\mathcal{C}_\Lambda(row)$ and $i=-N+1,...,N-1$.

Our claim is that the above equation, can be written as following expression (up to a multiplicative constant) which is based on graphical representation, see figs. (6) and (7),

$$
<--|\psi(k+\mathrm{i}m)|+->= c\sum_{\sigma\in\mathcal{C}_\psi}\alpha_c^{L(\sigma)}A_\psi(-1)(1+\eta_z\mathrm{i})(-1)^{L_z(\sigma)}, \qquad (162)
$$

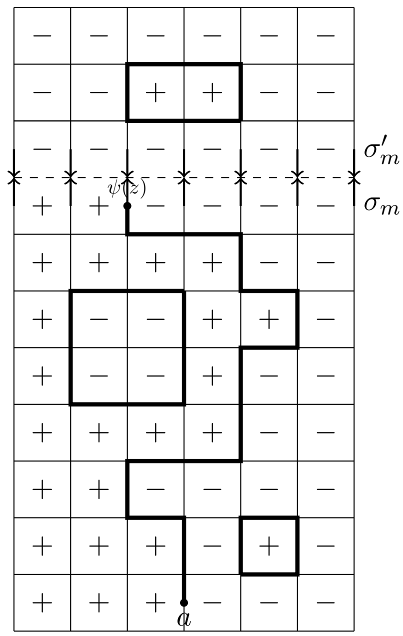

Figure 7: One-point fermion correlation function II

where the sum is over collections of dual edges in all configurations $\sigma \in \mathcal{C}_\psi$ with one fermion operator insertion, consisting of loops and a path from a boundary point to the fermion insertion point z, $\alpha_c = e^{-2\beta_c} = \sqrt{2} - 1$, $L(\sigma)$ is the number of the edges in a configuration σ, $L_z(\sigma)$ is the number of the vertical edges in row m on the right hand side of the z in the configuration σ, η_z is $+1$ or -1 according to two different possibilities in picture; a path can end at fermion insertion either from above (-1) or from below $(+1)$. This result is one of the important results of this thesis. Therefore, we will give a detailed proof of that. The proof is based on graphical representation. Let prescribe definite rules for graphical representation of the expansion of fermions correlation functions.

For any given configuration, we draw edges on the dual graph which follow these rules: 1) draw a vertical edge between each two subsequent sites with opposite spins and 2) start from the right hand side of each row, draw a horizontal edge when the parity of vertical edges changes. In other words, the term in the sum is non-zero when all boundary spins agree for the each V, i.e.

$$
\begin{aligned}
-1 &= (\sigma_{-N+1})_M = \cdots = (\sigma_{N-1})_M = (\sigma'_{N-m})_M \\
+1 &= (\sigma_{-N+1})_{-M} = \cdots = (\sigma_{N-m})_{-M} \\
-1 &= (\sigma'_{N-m})_{-M} = (\sigma_{N-m+1})_{-M} = \cdots = (\sigma_{N-1})_{-M},
\end{aligned}
$$

where $(\sigma_i)_j$ represents the spin on the $j-th$ column of the $i-th$ row of the lattice. Let us represent such a matrix element pictorially by drawing in row r a vertical line at between any j and $j+1$ with opposite spins, $(\sigma_r)_j \neq (\sigma_r)_{j+1}$ and between rows r and $r+1$ a horizontal line at position j if $(\sigma_r)_j \neq (\sigma_{r+1})_j$. Moreover, at vertical position m we split things in an upper and lower half, and in the upper half use σ'_m instead, and in the top and bottom rows we only use half a row. Note that the parity of lines coming to each point is even, with the exceptions of the boundary point $k' - iN$ where the boundary spin changes, and the point $k + im$, where we inserted the fermion. Following these rules, in any configuration or picture, V_1's count the number of horizontal edges, L_h, in picture and we have

$$\begin{aligned}(V_1)_{\sigma_i,\sigma'_i} &= \delta_{\sigma_i,\sigma'_i} \exp\left(\beta \sum_{j=-M}^{M-1} \sigma_j\sigma_{j+1}\right) \\ &= \delta_{\sigma_i,\sigma'_i} \exp(2M\beta)\exp(-2\beta L_h).\end{aligned}$$

In a similar fashion, one can show that V_2's count the number of vertical edges, L_v, in the picture, and we have

$$(V_2)_{\sigma_i,\sigma_{i+1}} = \exp((2M+1)\beta)\exp(-2\beta L_v).$$

Therefore, at critical temperature, using the relation, $\alpha_c = \exp(-2\beta_c)$, we find that, the V factors in low-temperature expansion of fermion correlation functions give rise to the factor α_c^L where $L = L_v + L_h$.

The second part of the proof is based on derivation of the contribution of $\psi_{k_{\sigma'_m,\sigma_m}}$ factor in low-temperature expansion. Using the relation, $\psi_k = A_\psi(p_k + q_k)$, and spin representations of p_k and q_k in eqs. (101) and (102) we obtain

$$\psi_k = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \otimes ... \otimes \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \otimes \left(A_\psi \begin{bmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{bmatrix} + A_\psi \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right) \otimes 1 \otimes ... \otimes 1.$$

Using this representation, we observe that the matrix element
$< \sigma'_m|\psi_k|\sigma_m >$ is non-zero only if

$$\begin{cases}(\sigma'_m)_j = & -(\sigma_m)_j \qquad \text{for } j < k\\ (\sigma'_m)_j = & (\sigma_m)_j \qquad \text{for } j > k\end{cases},$$

where j is the column index. As we mentioned, in order to derive the contribution of $\psi_{k_{\sigma'_m,\sigma_m}}$ in graphical representation, we use the relation $\psi_k = A_\psi(p_k + q_k)$, and representations of p_k and q_k. First, notice that there is a vertical edge ending at the mid-point k, because fermion operator ψ_k change the signs of spins on left of its insertion point k and keep the signs of spins on right of its insertion point k. Therefore, there is a vertical edge ending at point k either from above or from below. Also notice that when we have a vertical edge from below, then there is a change of spin signs at k in row configuration $\sigma_m = e^m_\sigma$, see fig. (7), and when we have a vertical edge from above, then there is no change of spin signs at column k in σ_m, see fig. (6). Then, we calculate explicitly the action of p_k and q_k on a row configuration by using the spin representations of p_k and q_k. Let us calculate complex factors of the whole expression. If there is a spin up (down) in the right hand side of k, in $k + \frac{1}{2}$, then p_k gives the $(+1)$ factor for spin up and (-1) factor for spin down. Similarly, q_k gives $(+\mathrm{i})$ for spin up and $(-\mathrm{i})$ for spin down located in the left hand side of k, in $k - \frac{1}{2}$. If we fix minus spins on the right boundary, it is not difficult to see that the actions of p_k and q_k on a row configuration give $(-1)(-1)^{L_z}$ and $(\eta_z \mathrm{i})(-1)(-1)^{L_z}$ factors, respectively. Finally, we can obtain that the factor from the matrix elements of ψ_k is

$$(\psi_k)_{\sigma'_m,\sigma_m} = A_\psi\,(-1)(-1)^{L_z}\,(1+\eta_z\mathrm{i}) = A_\psi\,(-1)(-1)^{L_z}\,(1+\mathrm{i})^{\eta_z}.$$

These arguments lead to the eq. (162).

After introducing the one-point fermion correlation functions, it would be natural to ask about higher-point fermion correlation functions and also spin-fermion correlation functions.

Consider two-point fermion-fermion correlation functions and its expansion. Similar to expansion of fermion correlation function, we have following expression which is based on graphical representation of the low-temperature expansion,

$$\begin{aligned}< ++|\psi(k+\mathrm{i}m)\psi(k'+\mathrm{i}m')|++ >= c \sum_{\sigma\in\mathcal{C}_{\psi\psi}} \quad & \alpha_c^{L(\sigma)} A_\psi^2 (1+\eta_z\mathrm{i})(-1)^{L_z(\sigma)} \\ & (1+\eta_{z'}\mathrm{i})(-1)^{L_{z'}(\sigma)}.\end{aligned} \tag{163}$$

where the sum is over collections of dual edges in all configurations $\sigma \in \mathcal{C}_{\psi\bar{\psi}}$ with plus boundary conditions and two fermion operators insertions, these configurations consist of loops and a path between $z = k + \mathrm{i}m$ and $z' = k' + \mathrm{i}m'$ and $L(\sigma)$, $L_z(\sigma)$ and $L_{z'}(\sigma)$ are defined in a similar way as above.

Similar calculations for fermion-antifermion correlations, lead to

$$< + + |\psi(k + \mathrm{i}m)\bar{\psi}(k' + \mathrm{i}m')| + + >= c \sum_{\sigma \in \mathcal{C}_{\psi\bar{\psi}}} \alpha_c^{L(\sigma)} A_\psi A_{\bar{\psi}} (1 + \eta_z \mathrm{i})(-1)^{L_z(\sigma)} (1 - \eta_{z'}\mathrm{i})(-1)^{L_{z'(\sigma)}}, \tag{164}$$

As a remark, notice that one-point fermion correlation functions can be considered as an special case of the two-point correlation functions, because the one-point correlation functions with the boundary state which has one boundary change can be seen as the two-point correlation functions when one of the fermion operators is inserted on the boundary state in the bottom of the lattice.

Expansions of multi-point fermion correlations and Wick's formula

The above results of two-point correlation functions can be generalized straightforwardly. The general idea of this section is that any $2n-$point correlation functions of the fermions and anti-fermions can be reduced to the sum of the products of the two-point correlation functions by using the lattice version of the Wick's formula.

The low temperature expansion for the $2n-$point fermion correlation functions is a straightforward generalization of the two-point results,

$$< + + |\psi(z_1)...\psi(z_{2n})| + + >= c \sum_{\sigma \in \mathcal{C}_{\psi 2n}} \alpha_c^{L(\sigma)} A_\psi^{2n} \prod_{j=1}^{2n} (1 + \eta_j \mathrm{i})(-1)^{L_{z_j}(\sigma)}, \tag{165}$$

where the sum is over collections of dual edges in all configurations $\sigma \in \mathcal{C}_{\psi^{2n}}$ with $2n$ fermion operators insertions, consisting of loops and n paths between any two fermions and $\eta_j = \pm 1$ depending on the choice, starting or ending at point z_j either from above or below.

In general, one can obtain a Pfaffian formula for $2n$-point correlation functions of fermions ψ, as follow

$$\frac{1}{Z} < + + |\psi(z_1)...\psi(z_{2n})| + + >= Pf\left(\left[\frac{1}{Z} < + + |\psi(z_i)\psi(z_j)| + + >\right]_{i,j=1}^{2n}\right), \tag{166}$$

where $Z =< ++|++ >$. Here we give a physical justification and sketch the proof of Pfaffian formula. The proofs of the Pfaffian formula or lattice fermionic Wick's formula is based on a special choice of isotropic splitting of the Fock space. In simple words, we define our polarization of the Fock space in such a way that the $| + + >$ state is the vacuum of the lattice theory meaning that it is annihilated by the annihilation operator. Then, notice that the fermion operator can be written as a linear expansion of annihilation and creation operators. The rest of the proof is just the same as the proof of the fermionic Wick's formula is quantum field theory. The exact derivation of the above Pfaffian formula is given in the proof of the theorem (23) in [HKZ12].

Expansions of spin-fermion correlations

In a similar way, the low-temperature expansion of spin-fermion correlation functions $\frac{1}{Z_\sigma} < --|\psi(z)\hat{\sigma}(z')|+- >$ with $Z_\sigma =< + + |\hat{\sigma}(z')| + + >$ can be obtained as follow:

$$< - - |\psi(z)\hat{\sigma}(z')| + - >= c \sum_{\sigma \in \mathcal{C}_{\psi\hat{\sigma}}} \alpha_c^{L(\sigma)} A_\psi (1 + \eta_z \mathrm{i})(-1)^{L_z(\sigma)}(-1)^{L_{z'}(\sigma)}, \tag{167}$$

where the sum is over collections of dual edges in all configurations $\sigma \in \mathcal{C}_{\psi\hat{\sigma}}$ with one fermion and one spin operators insertions, consisting of loops surrounding spin operator etc. and a path between a point on the boundary to the fermion insertion point z, a midpoint of horizontal edge, $\eta_z = \pm 1$ respectively in the cases that the path starting from the boundary point ends at fermion from above or below, z' is a point on a vertices of the lattice and $L_{z'}(\sigma)$ is the number of vertical edges on the right hand side of the point z' in the same row. In order to prove this expression, we need to evaluate the contribution of $\hat{\sigma}(z')$ in low-temperature expansion. It is easy to see that the contribution of spin operator is $(-1)(-1)^{L_{z'}(\sigma)}$ factor. Moreover, from previous sections, it is clear that the contribution of fermion $\psi(z)$ is the $(-1)(1+\eta_z \mathrm{i})(-1)^{L_z(\sigma)}$.

The spin-fermion correlation functions can be easily generalized to spin-fermion correlation functions with arbitrary number of spin operators

$$< - - |\psi(z) \prod_{s=1}^{S} \hat{\sigma}(z'_s)| + - >= c \sum_{\sigma \in \mathcal{C}_{\psi\hat{\sigma}^s}} \alpha_c^{L(\sigma)} A_\psi (-1)(1+\eta_z \mathrm{i})(-1)^{L_z(\sigma)} \prod_{s=1}^{S} (-1)(-1)^{L_{z'_s}(\sigma)}. \quad (168)$$

where the sum is over collections of dual edges in all configurations $\sigma \in \mathcal{C}_{\psi\hat{\sigma}^s}$ with one fermion operator and S spin operators insertions, consisting of loops and a path between a point on the boundary to the fermion insertion point z, a midpoint of horizontal edge, $\eta_z = \pm 1$ is defined as above, z'_s is a point on a vertices of the lattice and $L_{z'_s}(\sigma)$ is the number of vertical edges on the right hand side of point z'_s in the same row.

In the following we describe the relations between the definitions and properties of the fermions and their correlation functions to rigorously defined mathematical objects, s-holomorphic functions and parafermionic observables.

Smirnov observables & one-point Fermion correlations

The s-holomorphic winding observable of Smirnov, [Smi06], is crucial in the proof of convergence of Ising spin-interfaces to chordal SLE_3, and in fact it is the low temperature expansion of a one-point correlation function of the fermion ψ with one boundary change at the bottom boundary. However, s-holomorphic observables generalize this low-temperature expansion to different domains.

By comparing the formulas (162) and (133), it is the matter of combinatorics to find the relation between the one-point fermion correlations at critical temperature and s-holomorphic observables of Smirnov $F_a(z)$ as

$$\frac{1}{Z} < - - |\psi(z)| + - >= \sqrt{2}\lambda^3 A_\psi F_a(z), \quad (169)$$

where $Z =< + + | + + >$ is the partition function.

Sketch of the proof. Comparing two expressions (162) and (133), one can match the complex factors in two equations. On the one hand, the fermion contribution in the low-temperature expansion of $< - - |\psi(z)| + - >$ is the factor $\sqrt{2} A_\psi (-1) \lambda^{\eta_z} (-1)^{L_z(\sigma)}$. On the other hand, complex factor in Smirnov observable $F_a(z)$ is $e^{\frac{-\mathrm{i}}{2} w(\sigma)}$. It is basically a matter of combinatorics to check that $(-1)\lambda^{\eta_z}(-1)^{L_z(\sigma)} = \lambda^3 e^{\frac{-\mathrm{i}}{2} w(\sigma)}$. Combinatorics in the Smirnov observables consist of all the possibilities that a path starting upward from the point a and ending at z can have. This includes paths which turn either clock-wise or anti clock-wise and ending at z either from above or from below. This possibilities make differences in the winding of the paths and consequently in the computation of the observable and fermion correlation function. In all the cases, the mentioned equality is checked and is valid.

Energy observables & two-point fermion correlations

The s-holomorphic winding observable of [HoSm10b] is used for the derivation of the scaling limit of energy density of the critical Ising model. We show that it is a linear combination of two-point correlation functions of fermions ψ and $\bar{\psi}$ with constant boundary conditions. In order to find the relation between fermion-fermion correlations $< ++ |\psi(z)\psi(z')| ++ >$ and the parafermionic observables we have defined s-holomorphic functions, [HKZ12],

$$\begin{aligned} F^{\uparrow}_{z'}(z) &= \frac{1}{3}\sum_{\sigma\in\mathcal{C}_{z'\uparrow}} \alpha_c^{L(\sigma)} e^{-\frac{\mathrm{i}}{2}w(z'\rightsquigarrow z)}, \\ F^{\downarrow}_{z'}(z) &= \frac{1}{3}\sum_{\sigma\in\mathcal{C}_{z'\downarrow}} \alpha_c^{L(\sigma)} e^{-\frac{\mathrm{i}}{2}w(z'\rightsquigarrow z)}, \end{aligned} \tag{170}$$

where in $F^{\uparrow}_{z'}(z)$ ($F^{\downarrow}_{z'}(z)$) the paths starts at z' upward (downward).

In a slightly different notation, it has been shown in theorem (22) in [HKZ12] that the relations between two-point fermion correlations and the defined energy-density observables are

$$\frac{1}{Z} < ++ |\psi(z)\bar{\psi}(z')| ++ >= 2\mathrm{i}A_{\psi}A_{\bar{\psi}}(F^{\uparrow}_{z'}(z) + F^{\downarrow}_{z'}(z)), \tag{171}$$

and

$$\frac{1}{Z} < ++ |\psi(z)\psi(z')| ++ >= 2A^2_{\psi}(F^{\uparrow}_{z'}(z) - F^{\downarrow}_{z'}(z)). \tag{172}$$

For completeness of this part we sketch the proof of the above results. In order to calculate for example the fermion-fermion correlations $< ++ |\psi(z)\psi(z')| ++ >$ in terms of s-holomorphic observables we have to slightly change the derivation presented in previous part. The difference lies in the fact that in the fermion-fermion correlations there is no preferred direction for paths between z and z', because we have fermions at both z and z'. Therefore, we have to make a choice. We assume that all the paths starts at z' and ends at z. With this assumption we can compare the fermion-fermion correlations with the s-holomorphic functions defined above.

Let us start with fermion-antifermion correlation function. In fact, we can imagine that the configurations which have paths starting at $\bar{\psi}(z')$ from above and ending at $\psi(z)$ from either above or below, correspond to $F^{\uparrow}_{z'}(z)$. Equivalently, other configurations which have paths starting at $\bar{\psi}(z')$ from below and ending at $\psi(z)$ from either above or below, correspond to $F^{\downarrow}_{z'}(z)$. The equivalency between fermion-antifermion correlations and energy-density observables can be proved by careful consideration of combinatorial cases. One can imagine that there is four different possibilities corresponding to four different combinations of paths starting at $\bar{\psi}(z')$ from either above or below and ending at $\psi(z)$ from either above or below. Moreover, each path at z and z' can turn either clock-wise or anti-clock-wise.

Eventually, by comparing energy density observables expression and low temperature expansion of two-point correlation functions the equality $\mathrm{i}e^{\frac{-\mathrm{i}}{2}w(z'\rightsquigarrow z)} = \lambda^{\eta_z}(-1)^{L_z(\sigma)}\lambda^{-\eta_{z'}}(-1)^{L_{z'}(\sigma)}$ can be combinatorially checked separately in each of the four possibilities. Therefore, we obtain eq. (171).

About the fermion-fermion correlation functions, we follow the same graphical approach as in the case of fermion-antifermion correlations. Using the graphical representations, we compare the four possibilities and we obtain that the proportionality factor between s-holomorphic observables ($F^{\uparrow}_{z'}(z)$, $F^{\downarrow}_{z'}(z)$) and fermion-fermion correlation functions, eq. (172).

In general, any combination of multi-point fermion correlation functions of fermions ψ and $\bar{\psi}$, such as $\frac{1}{Z} < ++ |\psi(z_1)...\psi(z_{2m})| ++ >$ is related to a multi-point observable $F(z_1^{\eta_1}, ..., z_{2m}^{\eta_{2m}})$ used in [Hon10a] to derive energy correlations and boundary spin correlations.

SPINOR OBSERVABLES & SPIN-FERMION CORRELATIONS

Briefly stating, we obtained that the multi-valued s-holomorphic branching observables introduced in [ChIz11] and used in [CHI12] to prove conformal invariance of spin correlations, coincide with correlation functions of $\psi, \bar{\psi}$ and $\hat{\sigma}$, when an appropriate branch cut is used.

In the low-temperature expansion of spin-fermion correlations on a simply connected domain, we have two different kinds of configurations, first those which the path from a to z, fermion insertion point, crosses the row m' in the right hand side of z', spin insertion point, in odd number of times and second those which the path crosses even number of times. Thus, the spin-fermion correlation function can be written as

$$< - - |\psi(z)\hat{\sigma}(z')| + - >= c \sum_{\sigma \in \mathcal{C}_{\psi\hat{\sigma}}} \alpha_c^{L(\sigma)} A_\psi (1 + \eta_z \mathrm{i})(-1)^{L_z(\sigma)}(-1)^{l_{z'}(\sigma)}(-1)^{I_c(\sigma)}, \tag{173}$$

where $l_{z'}(\sigma)$ is the number of loops surrounding z' and $I_c(\sigma)$ is the indicator of crossing:

$$I_c(\sigma) = \begin{cases} 0 & \textit{if the path crosses even times} \\ 1 & \textit{if the path crosses odd times} \end{cases}. \tag{174}$$

This expression can be compared with the spinor observable in section (3.2.2). Similar to the previous cases, the fermion contribution in eqs. (167) or (173) leads to the winding term in eq. (139), modulo a complex factor. The spin contribution in both equations is the same and thus we obtain the relation between spin-fermion correlation functions and spinor observables as

$$\frac{1}{Z_\sigma} < - - |\psi(z)\hat{\sigma}(z')| + - >= -\sqrt{2}\lambda^3 A_\psi F_a^{spinor}(z, z'), \tag{175}$$

where $Z_\sigma =< + + |\hat{\sigma}(z')| + + >$ is the partition function in this case. We also observe that the spinor observables and fermion operators have the similar behavior at spin insertion.

PROOF OF THE PFAFFIAN FORMULA BY WICK'S FORMULA

The Pfaffian formula for the multi-point s-holomorphic functions can be proved by Riemann-Hilbert boundary value problem method [Hon10a]. Although, difficult combinatorial machinery has been used in the proof and moreover, for the case of massive s-holomorphic functions the proof is not known. However, by using the Wick's formula for fermion correlation functions and the fact that the multi-point correlation functions of fermions are linear combinations of multi-point s-holomorphic functions, we obtained easily the proof of the Pfaffian formulas for multi-point s-holomorphic functions in massless and massive cases, sections (4.4) and (4.5) in [HKZ12].

OPERATORS ON CAUCHY DATA SPACES

In section (5) in [HKZ12], we have constructed an algebraic framework which presents the geometric information of the domain in terms of an operator, called *Poincaré Steklov operator*. The scaling limit of this operator is well defined and therefore this can be considered as a starting point for an algebraic construction of rigorous scaling limit of the quantum states of the Ising model in s-holomorphic terms. The final goal would be a rigorous construction of well-defined quantum field theory for Ising model.

4.2 Chordal Schramm-Loewner evolution with $\kappa = 3$ and scaling limit of Ising free fermions

In this part, we summarize our results about the correspondence between fermionic CFT and SLE_3, [Za13]. The main idea is to construct a quantum field theory of Ising model which is a fermionic conformal field theory and study the relation between CFT and SLE_3 of the Ising model in an explicit manner. Consider the Ising model on a simply connected domain with boundaries. Assume plus/minus boundary conditions on the domain. Therefore, there are two boundary condition changing points on the boundary. At critical temperature the bulk of the lattice is divided into clusters with either plus or minus spins and a discrete path connecting two boundary points. In the scaling limit, when the lattice mesh tends to zero, these clusters are separated by random curves which we call them domain walls or interfaces, for a recent lecture on the subject see [Kon03]. Furthermore, a path starting at a marked point a on the boundary and separating minus spins from plus spins in the bulk of the domain will end at another boundary marked point b. This path is a random curve characterized by SLE curves.

The scaling limit of the Ising lattice interface is proved to be conformally invariant object and it is the Schramm-Loewner evolution, SLE_3, [CDHKS12]. Moreover, it has been shown in [Hon10a] that the scaling limit of the Ising model operators such as fermion etc. and their correlation functions converges to the free fermionic field theory, a CFT. This is a CFT with central charge $c = 1/2$ consisting of fermionic Fock space fields. In addition to Virasoro algebra, they respect also the Clifford algebra symmetry

The results that we are going to summarize consist of first, rigorous construction of the CFT for the scaling limit of the Ising fermions on bounded domains and second, a new explicit and rigorous realization and formulation of the known CFT/SLE correspondence in the case of Ising model. The heart of this part is the theorem that connects the VOA (as the Fock space of states) to the correlation functions of Fock space fields in domain D. This theorem is used to state rigorously the known results of CFT/SLE correspondence. We start with the definition of lattice fermion correlation functions in transfer matrix formalism. Then, by using the rigorous methods of discrete holomorphicity we find the scaling limit of the Ising fermion correlation functions on the half plane as well as any other domain. Following these result, the Fock space of fermionic states and fields, the operator product expansion, differential equations of the correlation functions in the conformal field theory can be constructed explicitly by using algebraic and analytic techniques. Our construction of the fermionic CFT is similar to the construction of CFT for Gaussian free fields in [KaMa11]. Then, the relation between fermionic CFT and SLE_3 is investigated, via an approach introduced in [BaBe06], at two levels: 1) the operator formalism in which we have the explicit form of the martingale generators in terms of Clifford vertex operator algebra and 2) the fermionic CFT correlation functions which are related to SLE_3 partition functions and martingale observables.

4.2.1 Ising fermionic CFT on bounded domains

In the Ising model, the lattice correlation functions of fermions can be defined and calculated by using the transfer matrix method as introduced in eq. (98). We define the two-point fermionic correlation functions as the scaling limit of the two-point lattice correlation functions of fermions

$$\lim_{\delta \to 0} \frac{1}{\delta} < + + |\psi(z)\psi(z')| + + >=< \psi(z)\psi(z') >_{Rectangle}, \tag{176}$$

where δ is the lattice mesh size and the limit is taken such that the length of the rectangle is kept fixed, the state $| + + >$ is defined in the paragraph "expansions of one-point and two-point fermion correlations" in section (4.1.2), in the left hand side of the above equation $z = k + \mathrm{i}m$ and $\psi(z) = V_M^{-m} \psi_k V_M^m$. Next step of the study is towards the scaling limit of the lattice correlation functions by using the methods of discrete holomorphicity. We summarize the results of our study about the free fermionic conformal field theories on bounded domains such as upper-half plane $\mathbb{H}$.

The two-point correlation functions of fermions on the upper-half plane are obtained rigorously by using first the relation between lattice correlation functions and s-holomorphic observables, found in [HKZ12] and second by discrete holomorphicity techniques that control the scaling limit of the s-holomorphic observables introduced in [HoSm10b] and [Hon10a]. For further descriptions of the derivations see section (2.3) in [Za13],

$$< \psi(z)\psi(w) >_{\mathbb{H}}= \frac{1}{z-w}, \quad < \bar{\psi}(\bar{z})\bar{\psi}(\bar{w}) >_{\mathbb{H}}= \frac{1}{\bar{z}-\bar{w}}, \quad < \psi(z)\bar{\psi}(\bar{w}) >_{\mathbb{H}}= \frac{1}{z-\bar{w}}. \tag{177}$$

In the following, we will use these results to check the characteristics and properties of fermionic CFT in an explicit and rigorous way similar to the introduced approach for Gaussian free fields in [KaMa11]. For further descriptions of the following results see sections (3.3)-(3.5) in [Za13]. The two-point correlation results can be transformed to any arbitrary domain D by a conformal map $g : D \to \mathbb{H}$. In general, in order to do the transformation of correlation functions, we need to know the transformation rules for primary field $\psi(z)$ and quasi-primary field $T(z)$ under a domain change, $g : D \to \mathbb{H}$,

$$\psi(z) = g'(z)^{\frac{1}{2}}\psi(g(z)), \quad T(z) = g'(z)^2 T(g(z)) + \frac{1}{24}S_g(z). \tag{178}$$

Then, the two-point correlation function of fermions on the domain D becomes

$$< \psi(z)\psi(w) >_D= \frac{g'(z)^{\frac{1}{2}}g'(w)^{\frac{1}{2}}}{g(z)-g(w)}. \tag{179}$$

Furthermore, by using the Taylor expansions of functions $g(z)$ and $g'(z)$ we obtained

$$< \psi(z)\psi(w) >_D= \frac{1}{z-w} + \frac{(z-w)}{12}S_g(w) + ... \tag{180}$$

This is an example of a general result which states that the singular part of the OPE of $\psi(z)$ and $T(z)$ are domain independent.

Given the two-point correlation functions of fermions $\psi(z)$, any $2n$-point correlation function on $\mathbb{H}$ can be obtained form the scaling limit of the multi-point lattice correlation as the Pfaffian formula,

$$< \psi(z_1)...\psi(z_{2n}) >_{\mathbb{H}}= Pf\left(\left[\frac{1}{z_i - z_j}\right]_{i,j=1}^{2n}\right). \tag{181}$$

Then by eq. (179), the $2n-$point correlation function of fermions on domain D is given by

$$< \psi(z_1)...\psi(z_{2n}) >_D= Pf\left(\left[\frac{\sqrt{g'(z_i)}\sqrt{g'(z_j)}}{g(z_i)-g(z_j)}\right]_{i,j=1}^{2n}\right). \tag{182}$$

It can be explicitly checked that the fermion correlation function on $\mathbb{H}$ in the Pfaffian form satisfies the Ward identity and the null field differential equation

$$< T(z)\psi(w_1)...\psi(w_n) >_{\mathbb{H}}= \sum_{i=1}^{N}\left[\frac{1/2}{(z-w_i)^2} + \frac{1}{z-w_i}\frac{\partial}{\partial w_i}\right] < \psi(w_1)...\psi(w_n) >_{\mathbb{H}}, \tag{183}$$

and

$$\left[\frac{3}{4}\frac{\partial^2}{\partial z^2} - \sum_{i=1}^{N}\left(\frac{1/2}{(z-w_i)^2} + \frac{1}{z-w_i}\frac{\partial}{\partial w_i}\right)\right] < \psi(z)\psi(w_1)...\psi(w_n) >_{\mathbb{H}}= 0, \tag{184}$$

where $\psi(w_i)$ are fermion primary fields with conformal dimension $h = \frac{1}{2}$ and $\psi(z)$ is a fermion primary field degenerate at level two. As we have seen, the null field differential equation plays an important role in CFT/SLE correspondence.

Moreover, by using the definition of the Virasoro field $T(z) = -\frac{1}{2} : \psi(z)\partial_z\psi(z) :$, Wick's theorem on domain D and Taylor expansion we found that the operator product expansion of the quasi-primary fields on the domain D are

$$\psi(z)\psi(w)|_D = \frac{1}{(z-w)} + reg(D),$$

$$T(z)\psi(w)|_D = \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial_w\psi(w)}{z-w} + \frac{3}{4}\partial_w^2\psi(w) + reg(D),$$

$$T(z)T(w)|_D = \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + reg(D), \tag{185}$$

where $reg(D)$ denotes the terms which do not diverge in the limit $z \to w$.

4.2.2 Clifford VOA and fermionic correlation functions

In this part, we will present a theorem which is the central result of this section. The known construction of the Clifford VOA, which is reviewed in sections (3.1) and (3.2) in [Za13], leads to the theorem which provides an explicit realization of fermionic Fock space of states and its relation to fermionic CFT and SLE_3. As we explained in section (3.2) in [Za13], the fermionic vertex operators are defined by the following formal power series

$$\psi(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k z^{-k-\frac{1}{2}}, \tag{186}$$

and

$$T(z) = \sum_{k\in\mathbb{Z}} L_k z^{-k-2}. \tag{187}$$

where L_k is Virasoro algebra generator. We have introduced a Fock space V of the fermionic states in section (3.2) in [Za13]. The highest weight vector of the Virasoro algebra is $\psi_{-\frac{1}{2}}|0>$ and basis vectors of the Fock space V are of the following form $\psi_{-k_n-\frac{1}{2}}\psi_{-k_{n-1}-\frac{1}{2}}...\psi_{-k_2-\frac{1}{2}}\psi_{-k_1-\frac{1}{2}}|0>$ for $k_n > k_{n-1} > ... > k_2 > k_1 > 0$. The fermionic Fock space $V = \bigoplus_{d\in\frac{1}{2}\mathbb{N}} V_d$ satisfies the vertex operator algebra axioms. This explicit Fock space of fermionic states in VOA can be used to present a concrete example of algebraic aspects of CFT/SLE correspondence in the case of Ising model. In order to proceed, by using a collection of results in CFT in domain D and Clifford VOA, we have proved rigorously the following theorem, in sections (3.2) and (3.5) in [Za13],

Theorem. There exists a unique mapping from $n-$th tensor power of V, the Fock space of states in VOA, to correlation functions of Fock space fields of the CFT in domain D; $\chi^{(D)}_{(z_1,...z_n)}(v_1 \otimes ... \otimes v_n) : V^{\otimes n} \to \mathbb{C}$, such that it satisfies the following properties:

1)

$$\chi^{(D)}_{(z_1,...,z_n)}(\psi \otimes ... \otimes \psi) = Pf\left(\left[\frac{\sqrt{g'(z_i)}\sqrt{g'(z_j)}}{g(z_i)-g(z_j)}\right]^n_{i,j=1}\right),$$

2)

$$\chi^{(D)}_{(z_1,...,z_n)}(v_1 \otimes ... \otimes L_{-1}v_i \otimes ... \otimes v_n) = \frac{\partial}{\partial z_i}\chi^{(D)}_{(z_1,...,z_n)}(v_1 \otimes ... \otimes v_n),$$

3)

$$\chi^{(D)}_{(z_1,...,z_n)}(v_1 \otimes ... \otimes v_m \otimes v_{m+1} \otimes ... \otimes v_n) =$$

$$\sum_{j=-\infty}^{N-1} \frac{1}{(z_m - z_{m+1})^{j+1}} \chi^{(D)}_{(z_1,...,\hat{z}_m, z_{m+1},...,z_n)}(v_1 \otimes ... \otimes v_m *_j v_{m+1} \otimes ... \otimes v_n), \tag{188}$$

where $v_m *_j v_{m+1}$ is the $j-$th OPE product in vertex operator algebra. In fact, the theorem provides a threefold connections between fermionic Fock space of states in VOA, the fermionic Fock space of fields and their correlators in CFT and martingale generators and observables in SLE_3.

4.2.3 CFT/SLE correspondence: Ising model

In section (2.5) we have discussed the fundamentals of CFT/SLE correspondence. In this part, we briefly discuss the relation and application of the previous constructions in VOA and CFT to the SLE_3. The main results include the explicit form, fermionic Pfaffian formula, for partition function of the chordal $n - SLE_3$ in section (4.1) in [Za13], explicit algebraic construction of the SLE_3 martingale generators in section (4.2) in [Za13] and SLE_3 martingale observables in terms of correlation functions of primary or descendant fermionic conformal fields in section (4.3) in [Za13].

SLE_3 partition function and fermionic Pfaffian formula

A partition function of a chordal $n - SLE_3$ can be obtained by considering the insertion of $2n$ fermion fields on the $2n$ boundary points, see CFT/SLE correspondence in [BaBe06]. Then, on the half plane the partition function is given by the $2n-$point correlation function of fermions which has the Pfaffian structure,

$$Z^{\mathbb{H}}_{n-SLE_3} = \chi^{(\mathbb{H})}_{(x_1,...,x_{2n})}(\psi \otimes ... \otimes \psi) =< \psi(x_1)...\psi(x_{2n}) >_{\mathbb{H}} = Pf([\frac{1}{x_i - x_j}]^{2n}_{i,j=1}). \tag{189}$$

From the Pfaffian structure of correlation functions one can check that $Z^{\mathbb{H}}_{n-SLE_3}$ is a positive smooth function which is covariant under Mӧbius transformation and it satisfies the null differential equation

$$\left[\frac{3}{4}\frac{\partial^2}{\partial x_i^2} + \sum_{l \neq i}\left[\frac{1}{x_l - x_i}\frac{\partial}{\partial x_l} - \frac{1/2}{(x_l - x_i)^2}\right]\right] Z^{\mathbb{H}}_{n-SLE_3} = 0. \tag{190}$$

This partition function can be used to define local martingale for multiple chordal SLE_3.

SLE_3 martingale generators and Clifford VOA

In order to find an explicit algebraic expression for SLE_3 martingale generators in terms of Clifford VOA, we need to check the level two singular vector condition for the fermionic Fock state $\psi_{-\frac{1}{2}}|0 >\in V$. As we observed, the fermion primary field degenerate at level two plays the role of boundary condition changing operator. Analogously, in the Clifford VOA language, the fermionic vertex operator has to play the same role. To show this in Clifford VOA language we use the fermionic representation of the Virasoro operator

$$L_m = -\frac{1}{2}\sum_{k \in \mathbb{Z}+\frac{1}{2}} (k + \frac{m}{2}) : \psi_{m+k}\psi_{-k} : + \frac{1}{16}\delta_m, \tag{191}$$

for $m \in \mathbb{Z}$, then by using the relation $[L_m, \psi_k] = -(\frac{1}{2}m + k)\psi_{m+k}$, it can be obtained that

$$(L_{-2} + \frac{3}{4}L^2_{-1})\psi_{-\frac{1}{2}}|0 >= 0, \tag{192}$$

which shows that the state $\psi_{-\frac{1}{2}}|0>$ is the highest weight state degenerate at level two. Then, by using the SLE martingale generator construction in [BaBe03], from eq. (69), the SLE_3 martingale generators can be written explicitly as vector valued local martingales

$$\frac{1}{Z}G_{f_t}\psi_{-\frac{1}{2}}|0>=\sum_{d\in\frac{1}{2}\mathbb{N}}M_d\in\bar{V}, \tag{193}$$

where Z is chordal SLE_3 partition function on the half plane and it is equal to one, $\bar{V}$ is the completion of V and for $\forall d$, $t\mapsto M_d(t)$ is a graded component of the local martingale of the chordal SLE_3.

SLE_3 MARTINGALE OBSERVABLES AND FERMIONIC CORRELATION FUNCTIONS

By using the theorem in section (4.2.2), we can explicitly construct the general Fock space fields and their correlation functions from VOA. Therefore, the CFT/SLE correspondence in the case of Ising model can be realized explicitly at the level of correlation functions via the theorem. Similarly, one can use the differential equations of correlation functions of primary or descendant fermionic fields on arbitrary domains to obtain SLE_3 martingale observables which satisfy correspondingly an stochastic differential equation with vanishing drift term. It has been shown in section (4.3) in [Za13] that a large collection of the chordal $n-SLE_3$ martingale observables are of the following explicit form

$$\frac{1}{Z^{H_t}_{n-SLE_3}}\chi^{(H_t)}_{(x_1,\ldots,x_{2n},z_1,\ldots,z_m)}((\psi\otimes\ldots\otimes\psi)\otimes(v_1\otimes\ldots\otimes v_m))=\frac{1}{Z^{H_t}_{n-SLE_3}}<\prod_{j=1}^{2n}\psi(x_j)\prod_{i=1}^{m}Y_i(z_i)>_{H_t}, \tag{194}$$

where H_t is the domain that $2n$ SLE_3 curves are removed from the half plane, $Z^{H_t}_{n-SLE_3}=\chi^{(H_t)}_{(x_1,\ldots,x_{2n})}(\psi\otimes\ldots\otimes\psi)=<\prod_{j=1}^{2n}\psi(x_j)>_{H_t}$, $\psi(x_j)$ is primary fermion field degenerate at level two inserted at boundary point x_j and $Y_i(z_i)$ is an arbitrary Fock space field at point z_i, corresponds to vector $v_i\in V$. The simplest case is the case $Y_i(z_i)=\psi(z_i)$ for all $i=1,\ldots,m$. In this case, the proof of the martingale property has been done rigorously in section (4.3) in [Za13].

PART II:

5 TRANSPORT PROPERTIES OF OPEN STRINGS AND BLACK HOLE MEMBRANE PARADIGM

In this chapter we review the standard subject of bosonic open string and a statistical framework for transport properties of a highly excited string introduced by Damour and Veneziano, [DaVe00]. Then we explain the powerful technique of linear response theory. Finally, we briefly review the thermodynamical aspects of the black holes and the membrane paradigm.

5.1 BOSONIC OPEN STRING THEORY

String theory is a candidate theory for the unification of general relativity and quantum mechanics. This unification plays an essential role in understanding the fundamental laws of Nature, [BBS07], [GSW87] and [Pol98]. The string theory consists of one-dimensional extended objects called strings. Size of a fundamental string is of the order of the planck-length. Although, string theory has its origins in describing strong force and QCD but it turned out that the theory is capable to construct a quantum theory that unifies the description of gravity and other fundamental forces of standard model.

In a naive sense, string theory can describe the unification because the strings with different modes of vibrations at low energies are different particles of matter and forces including gravity and gauge forces in nature. In spite of all the potentials and powerful mathematical techniques that string theory brings into the picture, it has rather strange features such as supersymmetry and extra dimensions which are not yet proven experimentally.

Let us start with a Lagrangian description of the string theory. A string lives on a $(d+1)$-dimensional curved space-time manifold M with metric $g_{\mu\nu}(x)$, $(\mu,\nu=0,...,d)$ and it sweeps a two-dimensional surface, called world-sheet Σ, during its time evolution in the space-time. The interactions in string theory can be described in terms of world-sheet topology. Therefore, strings can be described by the local sigma model action called *Polyakov action* of the world-sheet,

$$S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma \sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X), \tag{195}$$

where τ and σ are world-sheet coordinates, α' is a parameter proportional to the inverse of string tension and it measures the stringy effects, $X^\mu(\sigma,\tau):\Sigma\to M$ are the mappings from the world-sheet coordinates to background space-time coordinates, $\gamma_{\alpha\beta}(\sigma,\tau)$ is the auxiliary world-sheet metric, $\gamma = det\gamma_{\alpha\beta}$ and $g_{\mu\nu}$ is the space-time metric.

There are two types of strings, *closed strings* and *open strings* which are topologically, circles and line intervals, respectively. The boundary conditions for closed strings are periodical in σ and for open strings there are two possibilities, either they have to satisfy *Neumann* or *Dirichlet* boundary conditions.

Polyakov action in the Minkowski space background has specific symmetries:

• *Poincaré invariance*: invariance under global Poincaré transformations:

$$\delta X^\mu = a^\mu_\nu X^\nu + b^\mu, \quad \delta\gamma^{\alpha\beta}=0. \tag{196}$$

• *Re-parametrization invariance or diffeomorphisms*: invariance under local transformations of the world-sheet coordinates (σ,τ):

$$\sigma^\alpha \to f^\alpha(\sigma) = \sigma'^\alpha, \quad \gamma_{\alpha\beta}(\sigma) = \frac{\partial f^\rho}{\partial\sigma^\alpha}\frac{\partial f^\delta}{\partial\sigma^\beta}\gamma_{\rho\delta}(\sigma'). \tag{197}$$

• *Weyl invariance*: scale invariance of the action under local transformations:

$$\gamma_{\alpha\beta} \to e^{g(\sigma,\tau)}\gamma_{\alpha\beta}, \qquad \delta X^\mu = 0, \tag{198}$$

where $g(\sigma,\tau) : \Sigma \to \mathbb{R}$ is a positive function.

Re-parametrization invariance and Weyl invariance are local transformations, therefore we can make a choice and gauge-fix the induced metric. For example, we can make the following choice which is called conformal or unit gauge $\gamma_{\alpha\beta} = \eta_{\alpha\beta} = (-1, 1)$. In this gauge the action becomes

$$S_P = \frac{1}{4\pi\alpha'} \int d\tau d\sigma (\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu) g_{\mu\nu}(X), \tag{199}$$

where $\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}$ and $X'^\mu = \frac{\partial X^\mu}{\partial \sigma}$.

The classical equation of motion is derived by taking the variation of the action with respect to X^μ and it takes the form of wave equations:

$$\partial_\alpha \partial^\alpha X^\mu = 0, \qquad \text{or} \qquad (\frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2}) X^\mu = 0. \tag{200}$$

The solution of these equations can be written in a mode expansion. Depending on the imposed boundary conditions, the form of mode expansion changes, and for example for the Neumann boundary conditions, $X'_\mu = 0$ at $\sigma = 0, \pi$, it takes the following form:

$$X^\mu(\tau,\sigma) = \bar{x}^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\alpha_n^\mu}{n} e^{-in\tau} \cos(n\sigma), \tag{201}$$

where $\bar{x}^\mu$ is a central mass position, p^μ is the total string momentum and α_n^μ are oscillation modes.

The next step is to quantize the free theory of bosonic strings. Canonical quantization of the open strings leads to the following commutation relations:

$$[\bar{x}^\mu, p^\nu] = i\eta^{\mu\nu}, \qquad [\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}. \tag{202}$$

Moreover, there are different approaches in quantization of string theory such as path integral and BRST quantization, [Pol98].

Light-cone gauge quantization and Virasoro constraints

The equation of motion for auxiliary field $\gamma_{\alpha\beta}$ implies that the world-sheet stress tensor vanishes,

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{-\gamma}} \frac{\delta S_P}{\delta \gamma^{\alpha\beta}} = 0. \tag{203}$$

In the gauge $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$, the components of the stress tensor are

$$T_{01} = T_{10} = \dot{X} \cdot X' = 0, \qquad T_{00} = T_{11} = \frac{1}{4}(\dot{X}^2 + X'^2) = 0. \tag{204}$$

The Virasoro constrain for the bosonic string which is obtained from the vanishing of the components of stress tensor is

$$(\dot{X}^\mu \pm X'^\mu)^2 = 0. \tag{205}$$

At the classical level, the variations of the action with respect to X and γ give the variational equations, $\partial_\alpha \partial^\alpha X^\mu = 0$, $T_{\alpha\beta} = 0$. Analyzing these equations leads to the elimination of the two longitudinal vibrations of the $X(\Sigma)$ and leaving $d - 2$ transverse directions.

At the quantum level, the analysis of the constraints can be performed in different ways, such as direct imposing of the constraints on the physical states of the Fock space. In this approach, the constraints are imposed as an invariance conditions which select out an invariance subspace of the full Fock space.

Another approach is the method of light-cone gauge quantization. In this approach, the longitudinal degrees of freedom are eliminated at the classical level by first choosing a gauge in which the oscillation modes $\alpha_n^+ = 0$ and second, solving the longitudinal oscillator α_n^- in terms of the transverse oscillator by using the Virasoro constraints. Then, we can quantize the transverse directions. In the light cone gauge, the negative-norm states are manifestly excluded from the Fock space and thus it is possible to solve explicitly all the Virasoro conditions instead of imposing them as constraints. However, the Lorentz invariance is not manifest in the light-cone gauge quantization.

In order to perform light-cone gauge quantization, we use the light-cone coordinates of the space-time

$$X^{\pm} \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^d). \tag{206}$$

The mode expansion of the $X^{\pm}$ is trivially

$$X^{\pm}(\tau,\sigma) = \bar{x}^{\pm} + 2\alpha' p^{\pm}\tau + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\alpha_n^{\pm}}{n} e^{-in\tau}\cos(n\sigma). \tag{207}$$

Then, by using the residual gauge freedom of the action, we make a choice and take the non-covariant light-cone gauge as $X^+ = \bar{x}^+ + 2\alpha' p^+\tau$.

In the light-cone coordinates and by using the light-cone gauge, Virasoro constraint and its mode expansion can be written as

$$\dot{X}^- \pm X'^- = \frac{1}{4\alpha' p^+}(\dot{X}^i \pm X'^i)^2,$$

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}p^+}[\frac{1}{2}\sum_{i=1}^{d-2}\sum_{m\in\mathbb{Z}} : \alpha_{n-m}^i \alpha_m^i : -\delta_{n,0}]. \tag{208}$$

Moreover, the mass shell condition $M^2 = -p^\mu p_\mu$, can be obtained in terms of the level of the open string $N = \sum_{i=1}^{\infty} : \alpha_{-n}^i \alpha_{ni} :$ where :: is normal order product, the product with annihilation operators to the right. In fact, by using the $\alpha_0^\mu \equiv \sqrt{2\alpha'}p^\mu$ in the eq. (208), the mass shell condition can be obtained as

$$M^2 = \frac{1}{\alpha'}(N-1). \tag{209}$$

Finally, from the physical constraints such as non-negative norm states condition, it can be shown that the bosonic string theory as a quantum theory is consistent only in 26-dimensional space-time. In the spirit of CFT, it could be helpful to consider the bosonic open string theory on the flat Euclidean space $M = \mathbb{R}_E^{d+1}$. In this case, the Polyakov action for fixed metric γ defines a CFT of $d+1$ scalar fields X^μ with $c = d+1$. This is similar to the bosonic CFT that is explained in section (2.3).

5.1.1 STRING THEORY IN A CURVED BACKGROUND

In order to understand the dynamics of the open bosonic string in presence of external fields, we have to study the general action of the bosonic string with all possible couplings of the string with background fields. In general, the action is local in its dependence on X and other fields and is chosen to satisfy the diffeomorphism invariance $Diff(\Sigma)$, $Diff(M)$ and renormalizability conditions as a QFT. There are only certain background fields which are consistent with the above conditions and they are background massless fields: 1) metric tensor $g_{\mu\nu}$, 2) *Kalb-Ramond field* or 2-form B-field $B_{\mu\nu} \in \Omega^{(2)}(M)$ which is an anti-symmetric tensor field, 3) dilaton field $\Phi \in \Omega^{(0)}(M)$, with non-trivial vacuum expectation values. The action which include all these couplings is of the following form,

$$\begin{aligned} S_P = \quad & \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma \left[\sqrt{-\gamma}\gamma^{\alpha\beta}g_{\mu\nu}(X) + \epsilon^{\alpha\beta}B_{\mu\nu}(X)\right]\partial_\alpha X^\mu \partial_\beta X^\nu \\ & + \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{-\gamma}R^{(2)}\Phi(X), \end{aligned} \tag{210}$$

where $\epsilon^{\alpha\beta}$ is *Levi-Civita tensor* and $R^{(2)}$ is the *scalar curvature* of the metric $\gamma^{\alpha\beta}$.

In this study, we only consider a perturbation of the background metric from the flat metric and thus we set $g_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}(X)$, $B_{\mu\nu}(X) = 0$ and $\Phi(X) = 0$. Therefore, the action can be written in two separate terms as

$$\begin{aligned} S_P &= S_0 + S_1, \\ S_0 &= \frac{1}{4\pi\alpha'}\int d\tau d\sigma \left((\dot{X}^\mu)^2 - ({X'}^\mu)^2\right), \\ S_1 &= \frac{1}{4\pi\alpha'}\int d\tau d\sigma(\dot{X}^\mu\dot{X}^\nu - {X'}^\mu {X'}^\nu)h_{\mu\nu}(X). \end{aligned} \tag{211}$$

The response of the bosonic open string and its two-dimensional world-sheet to any change or deformation in the background geometry such as external perturbation in background metric is reflected in the space-time stress tensor

$$T^{\mu\nu} = -\frac{4\pi}{\sqrt{-g}}\frac{\delta S_P}{\delta g_{\mu\nu}}, \tag{212}$$

where $g = \det g_{\mu\nu}$.

5.1.2 Statistical theory of highly excited strings

As we mentioned, in order to study the transport properties of the open strings we need a statistical framework. Damour and Veneziano [DaVe00] developed a statistical formalism to study the size distribution and mass shift of a very massive highly excited single string state due to self-gravity. Moreover, they clarify the correspondence between string states and black holes, proposed by Susskind and by Horowitz and Polchinski.

It was known from 70's in the context of the dual resonance model that the spectrum of the string theory shows a large degeneracy of states which grow as an exponential of the mass. Moreover, the entropy of a string is proportional to the first power of mass, independent of the dimension of the space. However, there are natural questions such as how many massive string states have a certain size? To answer these questions, several approximations such as random walk model, self-gravitational model etc. have been proposed and different distribution functions in size of the ensemble of free string states with certain mass are introduced.

In order to systematically deal with these issues, a statistical approach has been initiated and developed for massive highly excited states.

In this section we review a grand-canonical ensemble for the states of bosonic strings, [DaVe00]. Let us start with a formal partition function of the system and the probability of the state $\{N_n^i\}$ in terms of levels of strings, in grand-canonical ensemble with fixed conjugate parameter β

$$Z(\beta) = tr(e^{-\beta N}) = \sum_{\{N_n^i\}} < \{N_n^i\}|e^{-\beta N}|\{N_n^i\} >= \sum_{\{N_n^i\}}\left(e^{-\beta N[N_n^i]}\right), \tag{213}$$

$$p[\{N_n^i\}] = Z^{-1}(\beta)e^{-\beta N[N_n^i]}, \tag{214}$$

where β is a formal conjugate to N, $N[N_n^i] = \sum_{n=1}^{\infty}\sum_{i=1}^{d-1} nN_n^i$ and $N_n^i = a_n^{i\dagger}a_n^i$ with $a_n^i = \frac{1}{\sqrt{n}}\alpha_n^i$ and $a_n^{i\dagger} = \frac{1}{\sqrt{n}}\alpha_{-n}^i$.

In this formulation, using the following formulas, $\log Z = \frac{(d-1)\pi^2}{6\beta}$ and $< N >_\beta = \bar{N} = -\frac{\partial \log Z(\beta)}{\partial\beta}$, the mass and entropy of the string can be obtained as

$$\begin{aligned} m^2 &= \frac{\bar{N}}{\alpha'} = \frac{(d-1)\pi^2}{6\beta^2\alpha'}, \\ S &= -tr(\rho\log\rho) = \log Z + \beta\bar{N} = 2\pi\sqrt{\frac{(d-1)\bar{N}}{6}}, \end{aligned} \tag{215}$$

where the density matrix is $\rho = Z(\beta)^{-1}e^{-\beta N}$ and the averages are given by $< A >_\beta = tr(\rho A)$.

Another ingredient that we need is the contraction between the oscillation modes which can be derived from the definition of the density matrix as

$$<: a_n^i (a_m^j)^\dagger :>_\beta = <: (a_m^j)^\dagger a_n^i :>_\beta = \frac{\delta^{ij}\delta_{nm}}{e^{\beta n} - 1}. \tag{216}$$

5.2 LINEAR RESPONSE THEORY AND KUBO'S FORMULA

In order to calculate the shear viscosity of the open string in section (6.1), we introduce briefly linear response theory in the following, [Cha00].

Applying an external disturbance to a physical system leads to a response from the system which is parameterized by a response function. The response function is the key concept towards the calculations of the transport coefficients of statistical systems via the Kubo's formula. Especially, we are interested in the hydrodynamical transport coefficients and in particular entropy and shear viscosity.

For simplicity, we will assume that the quantities are only time-dependent. When there is an external perturbation by weak fields $h_i(t)$, the Hamiltonian or the action of the system can be separated to free and perturbed parts

$$S = S_0 + \sum_i \int dt O_i(t) h_i(t), \tag{217}$$

where, S_0 is a free action, $O_i(t)$ is a conjugate operator of $h_i(t)$ and i denotes the number of external perturbation field. The expectation value $< O_i(t) >_h$ at time $(t > t_0)$ is defined by a time-dependent density matrix $\rho_h(t, t_0)$ in presence of external field as follow

$$< O_i(t) >_h = tr(\rho_h(t, t_0) O_i(t_0)). \tag{218}$$

The deviation of expectation value of an operator from its equilibrium value $\delta < O_i(t) > = < O_i(t) >_h - < O_i(t) >$, to the first order change in the external source $h_i(t)$ is given by

$$\delta < O_i(t) > \simeq \sum_j \int_{-\infty}^{\infty} dt' G^R(t - t') h_j(t'), \tag{219}$$

where $G^R(t - t')$ is the retarded Green function, called response function and it is given by

$$G^R(t - t') = \mathrm{i}\theta(t - t') < [O_i(t), O_j(t')] >, \tag{220}$$

where $\theta(t - t')$ is a step function.

The Fourier transform of the eq. (219) is

$$\delta < O_i(\omega) > = \sum_j G^R(\omega) h_j(\omega), \tag{221}$$

and in the low frequency limit, $\omega \to 0$, it takes the form

$$\delta < O_i(\omega) >= \sum_j (\chi_{\Re} + \mathrm{i}\chi_{\Im}\omega) h_j(\omega), \tag{222}$$

where $\chi_{\Re}$ and $\chi_{\Im}$ are the leading coefficients of real and imaginary parts of the response function in ω, respectively. The $\chi_{\Im}$ is called transport coefficient and it can be obtained in the low frequency limit by using the *Kubo's formula*:

$$\lim_{\omega \to 0} \frac{1}{\omega} \Im G^R(\omega) \simeq \chi_{\Im}. \tag{223}$$

In the following we will discuss the basics of black hole physics.

5.3 BLACK HOLES THERMODYNAMICS AND MEMBRANE PARADIGM

There are some controversial aspects of black holes such as information paradox, black hole radiation and membrane paradigm which are important unsolved problems in theoretical physics, [To97]. As it will be explained, the thermodynamics of the black holes possess a deep puzzle. Solutions to almost all of the mentioned unsolved problems need a complete statistical microscopic description of black holes which is not available now. However, string theory as a candidate theory of quantum gravity should in principle be able to describe black holes and their quantum behaviors and effects. In fact, string theory has been provided partial solutions to some aspects of black hole physics such as thermodynamical properties of classes of black holes.

Black holes are special singular solutions of Einstein equations, the fundamental equations of general relativity:

$$R_{\mu\nu} - g_{\mu\nu}(\Lambda - \frac{1}{2}R) = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{224}$$

where $R_{\mu\nu}$ is *Ricci curvature tensor*, R is *Ricci scalar*, $g_{\mu\nu}$ is metric, Λ is *cosmological constant*, $T_{\mu\nu}$ is stress tensor and G and c are Newton gravitational constant and speed of light, respectively. These are equations of motion obtained by variation of the Einstein-Hilbert action

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R, \tag{225}$$

where $g = \det g_{\mu\nu}$. These solutions describe the singular regions of the space-time which are causally disconnected from the asymptotic space at infinity. These singular regions have some boundaries that are shielding the singularity. This hypersurface is called "*event horizon*".

In three space dimensions, the space-time around a Schwarzschild black hole is described by the following metric in the spherical coordinate as

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{226}$$

This is a solution to the vacuum Einstein's equation, $R_{\mu\nu} = 0$. This region of the space-time is separated casually by the event horizon with the radius $R_S = \frac{2GM}{c^2}$.

There is a close relation between thermodynamics and black holes mechanics. In fact, general relativity implies that a black hole as a macroscopic object behaves like a thermodynamic object with an entropy which is called *Bekenstein-Hawking entropy*

$$S_{BH} = \frac{A_H}{4G}, \tag{227}$$

and temperature

$$T_H = \frac{\kappa}{2\pi}, \tag{228}$$

where A_H is the area of the horizon and $\kappa = \frac{c^2}{4GM}$ is the surface gravity at the horizon. Surface gravity on the event horizon as well as temperature are constant as the black hole is in the thermal equilibrium. Furthermore, the area law of the black holes $\delta A_{BH} \geq 0$ is consistent with the second law of the thermodynamics $\delta S \geq 0$.

In usual thermodynamics, we have two levels of descriptions for systems. Macroscopic and microscopic descriptions which the former is the coarse-grained description of the latter. Basically, these systems are described by a few macroscopic parameters such as energy, mass and volume. For each values of the parameters there are large numbers of microstates, Ω. This is a picture emerging from the atomic structure of the matter. Therefore, there is a natural basis for the degeneracies in the thermodynamical systems, which is stated as an entropy $S = k \ \log \Omega$, where k is the Boltzmann's constant. In the case of black holes, *"no-hair theorem"* restricts the characteristics of a black hole to its mass, charge and angular momentum. This theorem puts no space for any candidates for microstates, while the entropy of black hole necessarily implies the internal structure of the black hole which has numerous microscopic states. This leads to many subtleties in black hole physics.

MEMBRANE PARADIGM

In addition to the mentioned unsolved problems, there is a mystery about black holes, the *"membrane paradigm"*, [Mat10], [PaWi97] and [TPM86]. This is the toy model describing the quantum mechanical effects about black holes by using degrees of freedom on the surface of the extended event horizon. This is inspired by the conjecture that the dynamics of the bulk and internal states of the black holes might be described by the surface dynamics of the event horizon. This means that an effective theory of a fictitious electromechanical and hydrodynamical membrane can be used to understand the bulk dynamics of the black holes.

In other words, the membrane paradigm describes the fact that, in general relativity, transport coefficients such as shear viscosity and conductivity can be obtained on a stretched horizon of a black hole which is located near the event horizon. This means that there seems to be a fictitious viscous and conductive membrane living on the stretched horizon from the viewpoint of a distant observer. However, the microscopic interpretation of the membrane is not still clear.

ADM formalism, [ADM59], provides a framework in which the response of the black holes to external disturbances can be obtained from the Einstein equations. It can be derived in several different ways that the shear viscosity of the membrane is $\eta = \frac{1}{16\pi G}$, where G is the Newton gravitational constant, [TPM86].

In addition to hydrodynamical transport properties, there are other thermal and electrical transport properties and coefficients such as diffusion constant, electric conductivity, magnetic susceptibility etc. which a membrane can carry. For further investigations see [TPM86].

5.3.1 STRING THEORY/BLACK HOLE CORRESPONDENCE

The conjecture that the black holes with evolving mass in the radiation process, at some critical mass, transform to the string states and the Schwarzschild radius of black hole becomes of order of string scale was first initiated by Bowick et al [BSW87]. In fact, the value of the critical mass is given by the string mass scale m_s and its coupling constant g_s as $m_c \sim m_s g_s^{-2}$. Similarly, the correspondence between Schwarzschild black holes and fundamental strings with the fixed entropy and varying coupling g_s was conjectured by Susskind [Su93] and [Su94] as follow: as the coupling of the string increases, the size of the highly excited string state decreases and at some point becomes less than its Schwarzschild radius, R_S, and therefore string should transit to a black hole. On the other hand, when the coupling decreases, black hole states evolve to the string states as its size becomes less than the string scale, l_s. Horowitz and Polchinski [HoPo97] generalized this conjecture to charged black holes.

In general, the spectra of the black holes and single string agree and there is one-to-one correspondence between states of the Schwarzschild black holes and states of the fundamental strings. At some critical value of the string coupling $g_c^2 \sim m_s/m$, we have $R_S \sim l_s$, and thus both descriptions coincide and the black hole is described by the fundamental strings. In fact, the transition between a fundamental string and a black hole is smooth and it has been found that the Bekenstein-Hawking entropy of a Schwarzschild black hole can be reproduced by the entropy of a highly excited string covering the stretched horizon. As we mentioned, it has also been proposed that a highly excited string, when we increase the string coupling, becomes a black hole whose horizon radius is of the order of the fundamental string length scale l_s at the critical string coupling.

Although the mass dependence in the entropy formulas of the string and black holes are different, as for string we have $S_s \sim m/m_s$ and for black hole, $S_{BH} \sim (m/m_s)^2$ (in four dimensional space-time) they coincide at the critical coupling and critical mass, in Susskind and Bowick conjectures, respectively.

If black holes can be explained by string theory, the membrane paradigm must also be explained by string theory. Moreover, a microscopic description of the fictitious membrane might be understood through string theory which is closely related to black hole physics. In particular, from the perspective of the polymer representation of string/black hole correspondence [Kh99], we can obtain the shear viscosity of the stretched horizon from a highly excited string.

6 Shear viscosity of an open string and membrane paradigm: summary of results

In this chapter we use the statistical framework for the open strings and its relation to the unsolved problem of the membrane paradigm in black hole physics. In this study, we think of a bosonic string as a highly excited long string covering the fictitious membrane on the stretched horizon of a black hole. Then, by applying a time-dependent homogeneous background metric perturbation, string possess a viscosity. Thus, the highly excited states of a string can be seen as a viscous membrane.

Using the linear response theory, the shear viscosity of the highly excited string states can be calculated. Moreover, in the membrane paradigm, one can obtain the shear viscosity of the fictitious fluid on the stretched horizon of a black hole. Comparison of the obtained results about the transport coefficients in two approaches shows an agreement up to a numerical factor. The importance of this result is due to the recent calculations of the ratio of shear viscosity to entropy density by using the AdS/CFT correspondence, [SoSt07], and its applications in the membrane paradigm [IqLi08].

In summary, we describe the interpretation of the results as shear viscosity of fictitious fluid living on the horizon of the black hole in the contexts of string/black hole correspondence and membrane paradigm. In our study, [SaZa11], the transport coefficient that we are interested in, shear viscosity of a highly excited string, is obtained from the stress tensor by using Kubo's formula in linear response theory.

6.1 Entropy and shear viscosity of open strings

In order to calculate the shear viscosity, we adopt relations in section (5.2) in a way that the external source of the disturbance $h_{\mu\nu}$ is the perturbation of the geometry or the background metric from the flat space. The operator that is coupled with the metric tensor is the stress tensor and therefore the Kubo's relation (219) becomes

$$< T_{ij}(x^+) >= \int_{-\infty}^{\infty} dx'^+ \tilde{\chi}_{ij,kl}(x^+ - x'^+)' h^{kl}(x'^+). \tag{229}$$

Notice that the quantities are functions of light-cone coordinates. The response function $\tilde{\chi}_{ij,kl}(x^+ - x'^+)$ is given by

$$\tilde{\chi}_{ij,kl}(x^+ - x'^+) = 2\mathrm{i}\theta(x^+ - x'^+)\tilde{\chi''}_{ij,kl}(x^+ - x'^+). \tag{230}$$

where $\tilde{\chi''}_{ij,kl}(x^+ - x'^+) = \frac{V_d}{4} < [T_{ij}(x^+), T_{kl}(x'^+)] >$ and V_d is proportional to the volume that the string occupies. For details of the calculations see [SaZa11].

The Fourier transform of the Kubo's relation is

$$< T_{ij}(\omega) >= \chi_{ij,kl}(\omega) h^{kl}(\omega), \tag{231}$$

where $\chi_{ij,kl}(\omega) = \chi'_{ij,kl}(\omega) + \mathrm{i}\chi''_{ij,kl}(\omega)$ is the Fourier transform of the response function.

Then, the shear viscosity is given by the Kubo's formula

$$\eta = \lim_{\omega \to 0} \frac{1}{\omega} \Im G^R(\omega) = \lim_{\omega \to 0} \frac{2}{\omega} \chi''_{12,12}(\omega). \tag{232}$$

In [SaZa11] we have obtained the response function of the string by using the linear response theory as explained above with the help of mode expansion of the stress tensor and the introduced statistical framework for bosonic strings. In the low frequency limit the response function is obtained as

$$\chi''_{ij,kj}(\omega) \sim^{\omega \to 0} \frac{m}{4V_d} \sqrt{\frac{6\alpha'}{d-1}} \delta_{ij,kl} \omega, \tag{233}$$

where $\delta_{ij,kl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$.

Moreover, as we mentioned, the entropy of a highly excited string is

$$S \sim \sqrt{N} \sim m/m_s. \tag{234}$$

Before calculating the shear viscosity of the open bosonic string we need to understand the meaning of it. In calculation of the shear viscosity we think of a string as a polymer. Let us consider a polymeric liquid as an illustrating example [DoEd86]. It is known in polymer physics that the viscosity of a polymeric liquid is in general different from the viscosity of the original solvent because the stress tensor of the polymer itself is added to the stress tensor of the solvent. In other words, a polymer possess its own viscosity. In the same way, a string also possess its own viscosity because the string has its own stress tensor.

By using the Kubo's formula we obtain the shear viscosity, entropy density and ratio of the shear viscosity to entropy density of an open string as follow:

$$\eta = \sqrt{\frac{6}{d-1}}\frac{ml_s}{2V_d}, \qquad s = \frac{S}{V_d} = 2\pi\sqrt{\frac{d-1}{6}}\frac{ml_s}{V_d}, \qquad \frac{\eta}{s} = \frac{3}{2\pi c}. \tag{235}$$

6.2 Open string and membrane paradigm in black hole physics

In order to compare the obtained results about open strings and membrane paradigm, we have to introduce the longitudinally reduced string which produces quantities with appropriate dimensions, [SaZa11]. Then, the entropy and shear viscosity of the longitudinally reduced string on the stretched event horizon become of the same order as entropy and shear viscosity of the event horizon in the membrane paradigm. The main result of this chapter is that the shear viscosities for the reduced string covering the stretched horizon and for the membrane paradigm are equal up to a numerical factor as follow:

$$\eta_r = \sqrt{\frac{6}{d-1}}\frac{ml_s}{2V_{d-1}} \sim \frac{ml_s}{l_s^{d-1}} \sim \frac{1}{l_s^{d-1}g_c^2} \sim \frac{1}{16\pi G} = \eta_{BH}, \tag{236}$$

where V_{d-1} is proportional to the volume of the longitudinally reduced string and we have used the relation $G = g_s^2 l_s^{d-1}$.

Finally, in order to exactly match the ratio formula in membrane paradigm, $\frac{\eta_{BH}}{s_{BH}} = \frac{1}{4\pi}$ and in highly excited string formalism, $\frac{\eta_s}{s_s} = \frac{3}{2\pi c}$, we have to set $c = 6$. This value of central charge has been discussed in earlier works such as [HKRS96].

7 CONCLUSIONS

Different applications and realizations of conformal field theories have been provided predictive methods and techniques with fruitful insights and results in many physical systems. Our studies towards two of these applications represent the depth and power of CFT in studies of different physical systems.

I) Free fermions in the transfer matrix formalism of Ising model indirectly provide a rigorous mathematical framework to study the scaling limit of the Ising model as a quantum conformal field theory. The powerful analytic techniques from discrete holomorphicity and Riemann-Hilbert boundary value problems provide the rigorous results about Ising model. In fact, Ising free fermions are closely related to the precisely defined mathematical objects, s-holomorphic functions, which play the central role in controlling of the scaling limit of Ising model. We clarified these relations and then proposed a mathematically rigorous approach for exact scaling limit of the transfer matrix formalism in term of operators in Cauchy data spaces. Furthermore, in the continuum limit, free fermionic Fock space field theory and Clifford vertex operator algebra of the scaling limit of Ising model provides more concrete understanding of the SLE_3 and thus they explicitly exemplify the general statements in CFT/SLE correspondence.

II) Correctness and further applicability of the string theory/black hole correspondence has been approved in the context of membrane paradigm; we have shown that the string theory has correctly reproduce the shear viscosity of the membrane fictitious fluid.

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